Stochastic Mathematical Programs with complementarity constraints Using RQMC Methods

Research Report for Fieldhouse Fellowship

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1 Introduction

Mathematical program with complementarity constraints (MPCC) is a well known difficult optimization problem. MPCC is a challenge for computational science not only because of a non-convex feasible set and its non-smoothness, but also because of the high dimensionality. Furthermore, in the real world, uncertain factors should be allowed to be included. Such uncertainties result in stochastic optimization problems. Because such problems are even more difficult to handle with, good algorithms and designs are especially required to solve them.

2 Quasi-Monte Carlo and Randomized Quasi-Monte Carlo Method

Quasi-Monte Carlo (QMC) Methods use deterministic sequences with low discrepancy instead of a random one as in Monte Carlo (MC) methods. Applying randomization techniques to the low discrepancy sequences is to take advantage of the central limit theorem in order to estimate the error in QMC methods. The RQMC can also be viewed as a method which uses low discrepancy sequences as a variance reduction technique in MC methods.

A sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots$ of points in $[0,1)^s$ is called $(t,\, s)$ sequence in base b if for all integer $k > 0$ and $m > t$ the points set consisting of the bsx_n with $kb^m \leq n \leq (k+1)b^m$ is a $(t,\, m,\, s)$-net in base b. Refer to Niederreiter (1992) for definition of $(t,\, m,\, s)$-net. Scrambled $(t,\, s)$ sequences are developed by Owen. The basic idea is to scramble the digits of $(t,\, s)$ sequence, by using randomly chosen permutations from $b!$ permutations of $\{0, \cdots, b-1\}$ for the digits and in the same time preserving the low discrepancy property. For detailed method refer to Owen (1995) and Owen (1997). For special classes of functions, the variance of the quadrature rule based on scrambled nets can be as small as $O(n^{-3+\varepsilon})$.

Integration lattice is a low discrepancy design if its generator is well chosen. It has been widely used in multiple integration because of its simplicity and practicality for high dimensional problems during the past forty years (Sloan & Joe 1994). A d-dimensional integration lattice, L, is a set satisfying the following conditions

$$\mathbb{Z}^d \subseteq L \subset \mathbb{R}^d, \quad \boldsymbol{x}, \boldsymbol{y} \in L \Rightarrow \boldsymbol{x} + \boldsymbol{y} \in L.$$

The number of the node set of integration lattice, $\mathcal{P} := L \cap [0,1)^d$ is called cardinality and denoted by n. A lattice, which is generated only by one generator having no common factor with the cardinality n, is called rank-1 lattice.

Nowadays, integration lattice is also used in approximation (Kuo, Sloan & Wozniakowski 2005), (Zeng, Leung & Hickernell 2005) and partial differential equation (Li & Hickernell 2003). The error convergence rate is about $O(n^{-\beta+\varepsilon})$, where β is some number related with the smoothness of the function but not the dimension. To randomize the node set points of lattice $\mathcal{P}$, a random shift $\Delta \in [0,1)^d$ is added to node set points. The fractional parts of the new points produce the new copy of shifted lattice node set. Hence choosing M i.i.d random shifts would generate M independent copies.

3 Create SMPCC test cases

SMPCCs are optimization problems used in many fields such as data classification, traffic network, engineering design, energy market and etc. The main idea to solve SMPCCs is to reduce SMPCCs to MPCCs by sample average approximation method. The SAA methods are based on randomized Quasi-Monte Carlo (RQMC) sampling.

To demonstrate this method, we use the example 6.2 in Shapiro & Xu (2005) below .

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) = E_\omega\left(\sum_{i=1}^3 \left((x_i-1)^2 + y_i(\boldsymbol{x},\boldsymbol{\omega})\right)\right) \\ \text{subject to:} & x_1 \in [0,2], x_2 \in [0,2], x_3 \in [0,2]; \\ & 0 \geq (y_1,y_2,y_3) \perp (y_1 - x_1 + \omega_1, y_2 - x_2 + \omega_2, y_3 - x_3 + \omega_3) \geq 0; \end{array} \tag{1}$$

where $\omega_i, i = 1,2,3$ are independently uniformly distributed on $[0,1]$.

N realizations of the random vector $\boldsymbol{\omega}$ is generated by a QMC sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N$ with some randomization technique and then the expected value is approximated by the corresponding sample average. Moreover, the lower-order penalty method is applied to transform the complementary constrained optimization problem into unconstrained problems. Denote p as the penalty parameter and ϵ as the deviation from the zero for the complimentary constraints. In order to make the deviation not too far away from zero, the multiplication of p and ϵ is added to the objective function. Problem (1) is then reduced to the following optimization problem.

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) = 1/n \sum_{l=1}^n \left(\sum_{i=1}^3 \left((x_i-1)^2 + y_i(\boldsymbol{x},\boldsymbol{\omega}_l)\right)\right) + p\cdot\epsilon \\ \text{subject to:} & x_1 \in [0,2], x_2 \in [0,2], x_3 \in [0,2]; \\ & y_i \geq 0; y_i - x_i + \omega_i \geq 0; \\ & y_i(y_i - x_i + \omega_l) \leq \epsilon; \end{array} \tag{2}$$

The reduced problem (2) is solved by optimization solver Snopt (a software package for solving large-scale linear and nonlinear optimization problems). Repeat above procedure M times using M independent copies of the randomized QMC sequence and take the average of all the replications as the final result. The comparison of convergence rate using different sampling methods is shown in Figure 1. Apparently, using QMC sequence (lattice node set and (t, s) sequence) achieves much better convergence rate (approximately of $O(N^{-1})$) than just using MC methods (approximately of $O(N^{-1/2})$).

Beside the examples in Shapiro & Xu (2005), several models of stochastic mathematical programs with complementarity constraints are developed by

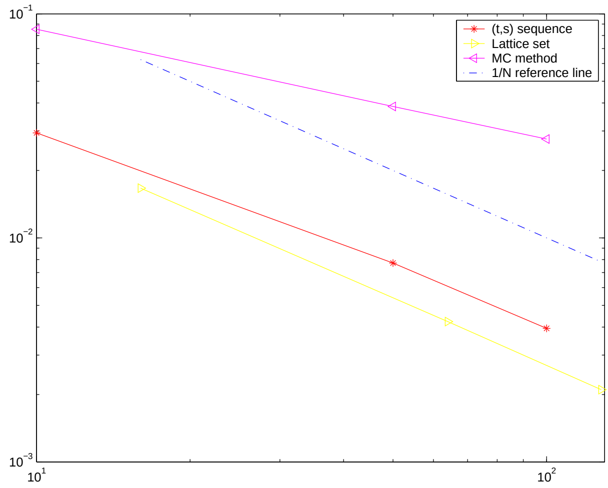

Figure 1: Error convergence result

us in this project. Those models include elastic models, based on Outrata, Kocvara & Zowe (1998), energy market model based on Li, Shahidehpour & Li (2006), toll pricing models based on Birbil, Gurkan & Listes (2004) and some others based on Lin, Chen & Fukushima (2005), Lin & Fukushima (2005), Kleywegt, Shapiro & De Mello (2001) and etc. We also created the sample sets of the randomized quasi-Monte Carlo sampling strategies which based on (t,s) sequence for all models and integration lattice for some of models. The entire approach are implemented by using AMPL (a modelling language for mathematical programming). The reduced MPCCs solved by SNOPT. AMPL and SNOPT is provided by Mathematical and Computer Science Division, Argonne National Laboratory. By the nature of the SAA methods, we use naturally parallelized algorithms to implement them using Blue Gene/L, the super computer in Argonne National Laboratory. The source codes for models can be downloaded through the link *www.iit.edu/∼zengxia/SMPCC/SMPCC.tar.gz* and the simulation results can be found at *www.iit.edu/∼zengxia/SMPCC/TABLE.html*.

4 Future Work

Up to date, the RQMC methods for solving SMPCCs are theoretically far away from well-developed. Our next goal is therefore to explore the theoretical result for the convergence rate. At the same time, we will develop smoothing algorithms for SMPCCs, because the QMC methods are much more efficient for smooth functions. Hence we can expect higher order of convergence of RQMC for smoothed SMPCCs in numerical simulations. This could lead to new methods that can be used to solve important and complex realistic problems.

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