Modeling Visual Transduction

Vena Pearl Boñgolan

Applied Mathematics Department,
Illinois Institute of Technology, Chicago,
Illinois

Department of Pharmacology, Vanderbilt University
Nashville, TN

Phototransduction

currently modeled as an uncoupled pair of deterministic, simple diffusion equations, interacting at the boundary conditions

$$G_t = C_g \Delta G, \tag{1}$$

$$C_t = C_c \Delta C, \tag{2}$$

where C_g is the diffusion coefficient for the first species, with concentration G, and C_c is the diffusion coefficient for the second species, with concentration C.

Boundary Conditions

At the boundary, the flux is
specified as:

$$C_g \frac{\partial G}{\partial z} = \kappa_1 F_1(C) \nabla G + \delta_j F_2(G), \tag{3}$$

$$C_c \frac{\partial C}{\partial n} = \kappa_2 F_3(C) - \kappa_3 F_4(G). \tag{4}$$

Motivation for Proposals

- currently modeled as an uncoupled pair of deterministic, simple diffusion equations, interacting at the boundary conditions

- we want a way of modeling uncertainty in the feedback mechanism, specially since they are coupled at the boundary conditions

- we may also try to model uncertainty in the diffusion process itself (stochastic heat equation), which is usually used to model discrete populations, e.g., chemical species.

- surfaces studied in biology are usually curved or irregularly shaped, so we might want to use a more general expression for the diffusion equation

- the complicated geometry of biological surfaces make for an apparent anisotropy in the diffusion process, even though it is molecularly isotropic.

Outline

- Noise in the Boundary
  - Past Research: Scalar Transport Equation
  - Past Research: Enstropy and Ergodicity
- Stochastic Heat Equation
- Diffusion on the Surface of a Sphere
- Anisotropic Diffusion
- Dynamics

Noise in the boundary conditions

- Use the present deterministic uncoupled system, and add noise on the boundary, similar to [**?**] and [**?**]. At present, we believe the theorem will work.
- Add boundary noise in the numerical simulations, similar to [**?**]

Outline

- Noise in the Boundary
  - **Past Research: Scalar Transport Equation**
  - Past Research: Enstropy and Ergodicity
- Stochastic Heat Equation
- Diffusion on the Surface of a Sphere
- Anisotropic Diffusion
- Dynamics

Past Research: Dynamics of Transport Under Random Fluxes on the Boundary

The impact of boundary noise on the dynamical evolution of the scalar transport equation in shear flows is studied, taking off from earlier studies in shear-flow dispersion in internal waves, a mechanism for horizontal mixing in the ocean. In particular, we model a gravity current evolving under an assumed shear-flow.

The transport equation is deterministic, with a noise term at the inlet *boundary*. This was motivated by observed seasonal fluctuations in some known sources of salty, dense water in the oceans, like the Red Sea overflow, as well as observed thermal and saline anomalies in the thermohaline circulation.

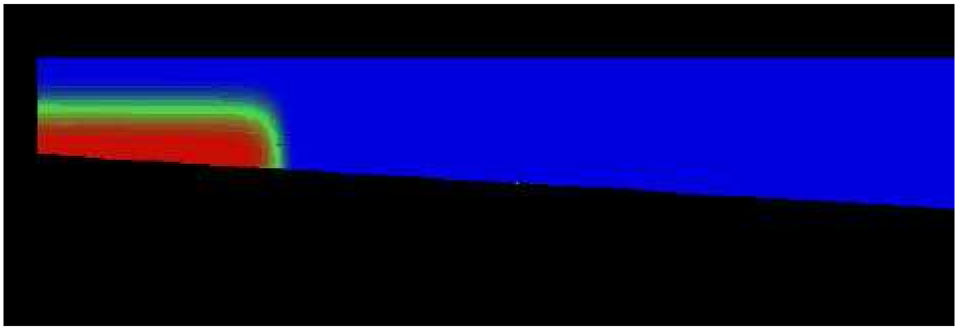

Figure 1: Color-coded density plot for temperature at time t=0; red is coldest, blue is least cold

Mathematical Formulation

The equation is

$$S_t + u \cdot \nabla S = \kappa \Delta S, \ (x, z) \in D \tag{5}$$

where $D = [0, 1] \times [0, 1]$, and $\kappa > 0$ is the salute diffusivity.

The Boundary Conditions

This linear transport equation is then supplemented with the following boundary conditions

$$\begin{aligned} \frac{\partial S}{\partial n} &= F(z) + \text{noise, on inlet boundary , x=0} \\ \frac{\partial S}{\partial n} &= 0 \text{ on rest of boundary} \end{aligned} \tag{6}$$

where $F(z)$ is the mean tracer or passive scalar flux. In the case of Wiener white noise, $\frac{\partial S}{\partial n} F(z) + \dot{W}_t$, where W_t is a Brownian motion defined in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

The Solution

A form for the solution is known, using a theorem of Da Prato and Zabczyk [**?**]. If we let A be the linear (in S) operator

$$A = -u \cdot \nabla - \kappa \Delta, \; (x, z) \in D (7)$$

then the equation has the form

$$S_t = AS \tag{8}$$

and we write the boundary conditions as a vector Y

$$Y = \begin{pmatrix} F(z) + \dot{W}_t \\ 0 \end{pmatrix}, \tag{9}$$

we may let operator γ define the boundary conditions as:

$$\gamma S = Y. \tag{10}$$

Here, we will just assume that A generates a C_0 semigroup of operators X(t), $t \geq 0$. A proper proof will require verification of the requirements of the Lumer-Philipps theorem.

If we designate as

$$\mathcal{N}(Y) = \phi \tag{11}$$

where

$$A(\lambda - \phi) = 0 \tag{12}$$

that is, $\mathcal{N}$ is the solution operator of the above eigenvalue problem, then we write the solution as:

$$S = X(t)S_0(x, z) + (\lambda - A)\int_0^t X(t-s)\mathcal{N}(Y)ds \tag{13}$$

Outline

- Noise in the Boundary
  - Past Research: Scalar Transport Equation
  - **Past Research: Enstropy and Ergodicity**
- Stochastic Heat Equation
- Diffusion on the Surface of a Sphere
- Anisotropic Diffusion
- Dynamics

Past Research: An Un-coupled System

The equations are

$$q_t + J(q, \psi) = \Delta q - Ra\partial_x S, \quad (14)$$

$$S_t + J(S, \psi) = \frac{1}{\mathrm{Pr}}\Delta S, \quad (15)$$

where $q(x, z, t) = -\Delta\psi$ is the vorticity in terms of stream function ψ, Pr is the Prandtl number and Ra is the Rayleigh number.

The Initial and Boundary Conditions

The boundary and initial conditions are:

$$q = 0, \text{on } \partial D, \tag{16}$$

$$\frac{\partial S}{\partial n} = 0, \text{ on } \partial D \backslash \{x = 0, \ 0 < z < 1\}, \tag{17}$$

$$\frac{\partial S}{\partial x} = F(z) + \dot{w}(z, t), \text{ on } \{x = 0, \ 0 < z < 1\}, \tag{18}$$

$$u(0) = u_0 = \begin{pmatrix} q_o \\ S_o \end{pmatrix}, \tag{19}$$

where n is the outward unit normal vector of ∂D. The system (**??**) consists of deterministic partial differential equations with stochastic Neumann boundary conditions.

Homogenization of Boundary Conditions ...

Note that we have a non-homogenous stochastic boundary condition for salinity S, so the first step is to homogenize this boundary condition.

To this end, we need an Ornstein-Uhlenbeck stochastic process solving the linear differential equation

$$\frac{d\eta_1}{dt} = \Delta\eta_1, \tag{20}$$

with the same boundary conditions as for S, and zero initial condition

$$\partial_x\eta_1(t,0,z,\omega) = F(z) + \dot{w}(z,t), \tag{21}$$

$$\partial_x\eta_1(t,1,z,\omega) = 0, \tag{22}$$

$$\partial_z\eta_1(t,x,0,\omega) = 0, \tag{23}$$

$$\partial_z\eta_1(t,x,1,\omega) = 0, \tag{24}$$

$$\eta_1(0,x,z,\omega) = 0, \tag{25}$$

$$\textstyle\int_D \eta_1 dD = 0. \tag{26}$$

Form of solution

We can write down the solution η_1 as

$$\eta_1(t,x,z,\omega) = (\lambda - \Delta) \int_0^t X(t-s)\mathcal{N}(\mathbf{Y})ds \tag{27}$$

where $\mathcal{N}$ is the solution operator to the elliptic boundary value problem $\Delta h - \lambda h = 0$ with the boundary conditions for h the same as η_1, that is $\partial h/\partial n = \mathbf{X}$ on ∂D with $\int_D h dD = 0$, where n is the unit outer normal vector to ∂D and $\mathbf{X}$ is

$$\mathbf{Y} = \begin{pmatrix} F(z) + \dot{w}(z,s) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here λ is chosen so that this elliptic boundary value problem has a unique solution. For our Neumann boundary conditions, we can choose λ to be any positive number, say $\lambda = 1$. Moreover, X(t) is a strongly continuous semigroup, symbolically, $e^{\Delta t}$, that is, the generator of $X(t)$ is Δ.

Outline

- Noise in the Boundary
  - Past Research: Scalar Transport Equation
  - Past Research: Enstropy and Ergodicity
- **Stochastic Heat Equation**
- Diffusion on the Surface of a Sphere
- Anisotropic Diffusion
- Dynamics

The Stochastic Heat Equation

$$G_t = C_g \Delta G + \dot{W}_t \qquad (28)$$

This has been used to model structured populations (where individuals can be grouped into discrete classes) in random environments [**?**]. Also related to the distribution of two-dimensional polymers in a random environment,[**?**].

What is the meaning in this case? Is it helpful in this case?

This model could then segue into a first-passage time problem, similar to our current research on the Morris-Lecar neural model.

Outline

- Noise in the Boundary
  - Past Research: Scalar Transport Equation
  - Past Research: Enstropy and Ergodicity
- Stochastic Heat Equation
- **Diffusion on the Surface of a Sphere**
- Anisotropic Diffusion
- Dynamics

The Heat Equation on the Surface of a Sphere

Currently, the diffusion equation being solved is the 'flat' case, as on a Euclidean plane, although the physical geometry is a cylinder.

$u_t = u_{xx} + u_{zz}$

There have been many investigations on solving the diffusion equation on curved or irregular surfaces, as in [**?**], [**?**],[**?**], and [**?**]. Several methods have emerged, from Monte-Carlo-based simulations to spherical harmonics and weighted fourier series representation.

On the surface of a sphere (r, θ, ϕ)

where $r = 1$ (a unit sphere), this would be

$$u_t = \csc^2 \phi u_{\theta\theta} + u_{\phi\phi} + \cot \phi u_\phi$$

Outline

- Noise in the Boundary
  - Past Research: Scalar Transport Equation
  - Past Research: Enstropy and Ergodicity
- Stochastic Heat Equation
- Diffusion on the Surface of a Sphere
- **Anisotropic Diffusion**
- Dynamics

Anisotropy in the Laplacian

The problem of anisotropy in the Laplacian (controversially) appears in oceanographic applications as a result of disparate length scales [**?**], and we see it again as an *apparent* anisotropy in biological applications as a result of differing membrane curvatures in different spatial directions. This has been studied in [**?**], [**?**] and [**?**].

Outline

- Noise in the Boundary
  - Past Research: Scalar Transport Equation
  - Past Research: Enstropy and Ergodicity
- Stochastic Heat Equation
- Diffusion on the Surface of a Sphere
- Anisotropic Diffusion
- **Dynamics**

Dynamics

What is the behavior of the system in phase-space? Are there recurrent motions? The deterministic Morris-Lecar neural model showed oscillations in [**?**], and its random dynamics were investigated in [**?**]. We did a theoretical investigation of recurrent motions for a Navier-Stokes system in [**?**].

Bibliography

[1] V.P. Boñgolan-Walsh, J. Duan, and T. Özgökmen. "Dynamics of Transport under Random Fluxes on the Boundary." to appear in *Communications in Nonlinear Science and Numerical Simulation, Elsevier* (2005)

[2] V.P. Boñgolan-Walsh, J. Duan, H. Gao, T. Özgökmen,P.F. Fischer, and T. Iliescu. "Enstrophy and Ergodicity of Gravity Currents." *IMA Volumes in Mathematics and its Applications, Springer-Verlag, 140, 61-78* (2004)

[3] G. Da Prato and J. Zabczyk. Evolution equations with white noise boundary conditions, *Stochastics and stochastics Reports*, Vol 42(1993), 167–182.

[4] L. Demetrius and V.M. Gundlach, *Evolutionary Dynamics in Random Environments (Stochastic Dynamics)*, H. Crauel and M. Gundlach, Eds., Springer, New York, 1999.

[5] L. Bertini and N. Cancrini.The Stochastic Heat Equation: Feynman-Kac Formula and Intermittence , *Pre-print 1032, Dipartimento di Fisica, Universita di Roma "La Sapienza", 1994.*

[6] M.M.G. Krishna, J. Samuel and S. Sinha. Brownian Motion on a Sphere:Distribution of Solid Angles. *J. Phys. A: Math. Gen.*, Vol 33 (2000), 5965–5971.

[7] M.M.G. Krishna, R. Das and N. Periasamy. Translational Diffusion of Flourescent Probes on a Sphere: Monte-Carlo Simulations, Theory and Flourescents Anisotropy Experiment. *Journal of Chemical Physics*, Vol 112, No. 19 (2000), 8502–8514.

[8] I.F. Sbalzarini, A. Hayer and P. Koumoutsakos. Simulations of (An)Isotropic Diffusion on Curved Biological Surfaces. *Biophysical Journal*, Vol 90 (2006), 878–885.

[9] M. K. Chung, Unified Cortical Surface Morphometry and its Application to Quantifying Amount of Gray Matter. *Technical Report No. 1122, Department of Statistics, U. of Wisconsin-Madison* (2006).

[10] V.P. Boñgolan-Walsh, J. Duan, P.F. Fischer, T. Özgökmen, and T. Iliescu. “Impact of Boundary Conditions on Entrainment and Transport in Gravity Currents.” to appear in *Journal of Applied Mathematical Modelling, Elsevier* (2005)

[11] M. Christensen. How to Simulate Anisotropic Diffusion Processes on Curved Surfaces. *Journal of Computational Physics*, Vol 201 (2004), 421–438.

[12] C. Morris and H. Lecar,Voltage Oscillations in the Barnacle Giant Muscle Fiber. *Biophys. Journal* **35**:193-213, (1981).

[13] T. Tateno and K. Pakdaman,Random Dynamics or the Morris-Lecar Neural Model. *Chaos* **14**:511-530, (2004).

[14] V.P. Boñgolan-Walsh, D. Cheban, and J. Duan. “Recurrent Motions in the Nonautonomous Navier-Stokes System.” *Discrete and Continuous Dynamical Systems: Series B, 3, 255-262* (2003)

[15] H.C. Tuckwell, *Stochastic Processes in the Neurosciences*, SIAM, Philadelphia, 1989.

[16] I. Goychuk and P. Hänggi,Ion Channel Gating:A First-passage Time Analysis of the Kramers Type. *PNAS* **99**:3552-3556, (2002).

[17] C. Chevalier and F. Debbasch,Multi-Scale Diffusion on 2D Nearly Flat Riemannian Manifolds.*Submitted to European Physics Letters?* (2006).