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Almost Periodic Solutions and Global Attractors of Non-autonomous Navier–Stokes Equations

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The article is devoted to the study of non-autonomous Navier–Stokes equations. First, the authors have proved that such systems admit compact global attractors. This problem is formulated and solved in the terms of general non-autonomous dynamical systems. Second, they have obtained conditions of convergence of non-autonomous Navier–Stokes equations. Third, a criterion for the existence of almost periodic (quasi periodic, almost automorphic, recurrent, pseudo recurrent) solutions of non-autonomous Navier–Stokes equations is given. Finally, the authors have derived a global averaging principle for non-autonomous Navier–Stokes equations.

KEY WORDS: Non-autonomous dynamical system; skew—product flow; global attractor; non-autonomous Navier–Stokes equation; almost periodic solutions; global averaging principle.
(Revised Version)

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1. INTRODUCTION

We consider the two-dimensional Navier–Stokes system

$$u' + q(t)\sum_{i=1}^{2} u_i \partial_i u = \nu \Delta u - \nabla p + \phi(t) \tag{1}$$

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$$\text{div } u = 0, \quad u|_{\partial D} = 0,$$

where D is an open bounded set with boundary $\partial D \in C^2$. This equation can be written in the following form

$$u' + Au + B(t)(u, u) = f(t) \tag{2}$$

on the corresponding Sobolev's space E, where $-A$ is a Stokes operator, $B(t)$ is a bilinear form satisfying the identity

$$Re\langle B(t)(u, v), w\rangle = -Re\langle B(t)(u, w), u\rangle \tag{3}$$

for all $t \in \mathbb{R}$ and $u, v, w \in E$, and f is forcing term.

In the work [11,13,17,18], a non-stationary Eq. (2), where f is a function of time $t \in \mathbb{R}$ is studied. It is shown that the equation with compact f (in particularly, almost periodic) admits a compact global attractor and also for small non-linear (bilinear) term it was proved the existence a unique almost periodic (quasi-periodic, periodic) solution of Eq. (2) if the forcing term f is almost periodic (quasi-periodic, periodic).

The aim of the present article is to study Eq. (2) in the case, when the bilinear form B, and the function f are non-stationary. The conditions under which a non-stationary equation of type (2) admits a compact global attractor are indicated.

The theorem of "partial" averaging on finite interval for ordinary differential equations it was proved in the work [15]. The works [13,17,18] are devoted to generalization of method of averaging for dissipative partial differential equations. We prove the theorem of "partial" averaging for non-autonomous Navier–Stokes equation (2) (i.e., the bilinear form and forcing term are non-stationaries).

Our paper is organized as follows:

In Section 2 we introduce a class of non-autonomous Navier–Stokes equations and establish its dissipativity (Theroem 2.8).

In Section 3 we prove that non-autonomous Navier–Stokes equations admit a compact global attractor (Theorem 3.8).

Section 4 is devoted to study of the problem of existence of almost periodic (quasi-periodic, recurrent, pseudo-recurrent) solutions of non-autonomous Navier–Stokes equations (Corollary 4.20) and we give the conditions of convergence of this equations (Theorem 4.17).

In Section 5 we prove the uniform averaging principle for the non-autonomous Navier–Stokes equations on the finite segment (Theorem 5.2).

Section 6 is devoted to prove the global averaging principle for non-autonomous Navier–Stokes equations on the semi-axis (Theorems 6.1, 6.3 and 6.7).

2. NON-AUTONOMOUS NAVIER–STOKES EQUATIONS

Some results from the theory of semigroups of linear operators [16] and PDEs [17, 26, 29] are collected below.

A closed operator A with domain $D(A)$ that is dense in a Banach space X is called a sectorial operator if for some $a \in \mathbb{R}$ and $\varphi \in (0, \frac{\pi}{2})$ the sector

$$S_{a,\varphi} := \{\lambda \in \mathbb{C}, \pi \geqslant |arg(\lambda - a)| \geqslant \varphi\} \tag{4}$$

is contained in the resolvent set and for $\lambda \in S_{a,\varphi}$

$$\|(\lambda I - A)^{-1}\|_{X \to X} \leqslant \frac{c}{|\lambda - a| + 1}. \tag{5}$$

For a sectorial operator A the analytic semigroup of linear bounded operators in X is defined and denoted by $e^{-At}, t \geqslant 0$.

Let A be a sectorial operator with $Re\ \sigma(A) > 0$. For $\alpha \in (0, 1)$ we define fractional powers of A as follows:

$$A^{\alpha} := (A^{-\alpha})^{-1}, \quad \text{where } A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-At}\ \mathrm{d}t.$$

The corresponding domains $D(A^{\alpha})$ are Banach spaces with norm given by

$$|\cdot|_{\alpha} := |\cdot|_{D(A^{\alpha})} = |A^{\alpha} \cdot |.$$

Theorem 2.1. *The following estimates are valid:*

$$\text{(i)} \qquad \|e^{-At}\|_{X \to X} \leqslant Ce^{-at}, \quad t \geqslant 0, \tag{6}$$

$$\text{(ii)} \qquad \|A^{\alpha} e^{-At}\|_{X \to X} \leqslant C_{\alpha} t^{-\alpha} e^{-at}, \quad t > 0. \tag{7}$$

Let Ω be a compact metric space, $\mathbb{R} = (-\infty, +\infty)$, $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on $\Omega, \mathcal{E}$ be a real or complex Hilbert space, $L(\mathcal{E})$ be the space of all linear forms on $\mathcal{E}, L^2(\mathcal{E})$ be the space of all bilinear continuous forms $B: \mathcal{E} \times \mathcal{E} \to \mathcal{F}$ and $C(\Omega, W)$ be a space of all continuous functions $f: \Omega \to W$ (W is some metric space), endowed with the topology of uniform convergence. Let us consider the equation

$$u' + Au + B(\omega t)(u, u) = f(\omega t) \tag{8}$$

($\omega \in \Omega$) where $\omega t := \sigma(t, \omega)$, $B \in C(\Omega, L^2(\mathcal{E}))$, $f \in C(\Omega, \mathcal{E})$ and A is a linear operator.

Below we will use some notions, denotations and results from [18]. Let Hilbert spaces E, F, X satisfy $E \subset F$; $E, F, X \subset \mathcal{E}$, each embedding being dense and continuous.

We further suppose that the linear operator A is densely defined in $\mathcal{E}$ and such that the linear equation

$$u' + Au = 0 \tag{9}$$

generates the c_0-semigroup of linear bounded operators

$$e^{-At} : \mathcal{E} \to \mathcal{E}, \qquad \varphi(t, x) := e^{-At} x,$$

which for $t > 0$ can be extended to the linear bounded operators from F to E satisfying the following estimates

$$\|e^{-At}\|_{E\to E} \leqslant K e^{-at}, \tag{10}$$

$$\|e^{-At}\|_{F\to E} \leqslant K t^{-\alpha_1} e^{-at}, \quad 0 \leqslant \alpha_1 < 1, \tag{11}$$

$$\|Ae^{-At}\|_{F\to E} \leqslant K t^{-\alpha_2} e^{-at}, \quad 0 \leqslant \alpha_2 < 2. \tag{12}$$

We also suppose that the following condition is satisfied

$$Ae^{At} = e^{At} A, \tag{13}$$

in the sense of $L(F, E) := \{A : F \to E | A \text{ is linear and bounded}\}$ equipped with the operational norm.

Bilinear form B. Denote by $L^2(E, F)$ the space of all bilinear bounded form $B : E \times E \to F$ with the norm

$$\|B\| := \sup\{|B(u, v)|_F : |u| \leqslant 1, |v| \leqslant 1\}.$$

Let $C(\Omega, L^2(E, F))$ be a space of all continuous mappings $B : \Omega \to L^2(E, F)$ and

$$C_B := \sup\{|B(\omega)(u, v,)|_F : \omega \in \Omega, |u| \leqslant 1, |v| \leqslant 1\},$$

then the mapping $F \colon \Omega \times E \to F(F(\omega, u) := B(\omega)(u, u))$ satisfies the following inequality

$$|B(\omega)(u_1, u_1) - B(\omega)(u_2, u_2)|_F \leqslant C_B(|u_1|_E + u_2|_E)|u_1 - u_2|_E \tag{14}$$

for all $u_1, u_2 \in E$.

From the inequality (14) follows that on every ball $B[0, R] := \{u \in E : |u| \leqslant R\}$ we have

$$|B(\omega)(u_1, u_1) - B(\omega)(u_2, u_2)|_F \leqslant 2C_B R |u_1 - u_2|_E \tag{15}$$

for all $u_1, u_2 \in E$.

Remark 2.2. The space of all the bilinear form $C(\Omega, L^2(E,F))$ is a Banach space with the norm $\|B\| := C_B$.

Function f. The external force $f : \Omega \to X$ is continuous, i.e., $f \in C(\Omega, X)$.

Operations e^{-At}. The operators $e^{-At}(t>0)$ can be extended to the linear bounded operators from X to E satisfying the estimates

$$\|e^{-At}\|_{X\to E} \leqslant Kt^{-\beta_1}e^{-at}, \quad 0 \leqslant \beta_1 < 1, \tag{16}$$

$$\|Ae^{-At}\|_{X\to E} \leqslant Kt^{-\beta_2}e^{-at}, \quad 0 \leqslant \beta_2 < 2, \tag{17}$$

and Eq. (13), this time in the sense of $L(X,E)$.
We suppose that the following conditions are fulfilled:

(i) there exists $\alpha > 0$ such that

$$Re\langle Au, u\rangle \geqslant \alpha |u|^2 \tag{18}$$

for all $u \in E$, where $|\cdot|$ is a norm in E;

(ii) $$Re\langle B(\omega)(u,v), w\rangle = -Re\langle B(\omega)(u,w), v\rangle \tag{19}$$

for every $u, v, w \in E$ and $\omega \in \Omega$.

Remark 2.3. (a) It follows from (19) that

$$Re\langle B(\omega)(u,v), v)\rangle = 0 \tag{20}$$

for every $u, v \in E$ and $\omega \in \Omega$.

(b) $$|B(\omega)(u,v)|_F \leqslant C_B|u|_E|v|_E \tag{21}$$

for all $u, v \in E$ and $w \in \Omega$, where $C_B = \sup\{|B(\omega)(u,v)|_F : \omega \in \Omega, u, v \in E,\ |u|_E \leqslant 1, \text{ and } |v|_E \leqslant 1\}$.

Eq. (8) with conditions (18) and (19) is called a non-autonomous Navier–Stokes equation. We will consider the mild solutions of Eq. (8), i.e., $u \in C([0,T], E)$ and satisfy the following integral equation

$$u(t) = e^{-At}x + \int_0^t e^{-A(t-s)}(-B(\omega s)(u(s), u(s)) + f(\omega s))\mathrm{d}s. \tag{22}$$

Theorem 2.4. *Let $x_0 \in E, r > 0$ and the conditions (10), (11) and (18) are fulfilled, then there exist positive numbers $\delta = \delta(x_0, r)$ and $T = T(x_0, r)$ such that Eq. (22) admits a unique solution $\varphi(t,x,\omega)$ $(x \in B[x_0,\delta] = \{x \in E \,|\, |x - x_0| \leqslant \delta\})$ defined on the interval $[0,T]$ with the conditions: $\varphi(0,x,\omega) = x$, $|\varphi(t,x,\omega) - x_0| \leqslant r$ for all $t \in [0,T]$ and the mapping $\varphi : [0,T] \times B[x_0,\delta] \times \Omega \to E((t,x,\omega) \to \varphi(t,x,\omega))$ is continuous.*

Proof. Let $x_0 \in E, r > 0, \delta > 0$ and $T > 0$. We consider a space $C_{x_0,r,\delta,T}$ of all continuous functions $\psi : [0,T] \times B[x_0,\delta] \times \Omega \to B[x_0,r]$ equipped with the distance

$$d(\psi_1,\psi_2) := \sup\{|\psi_1(t,x,\omega) - \psi(t,x,\omega)|_E \; : \; 0 \leqslant t \leqslant T, x \in B[x_0,\delta], \omega \in \Omega\}$$

is a complete metric space.

We define the operator Φ acting onto $C_{x_0,r,\delta T}$ by the equality

$$(\Phi\psi)(t,x,\omega) = e^{-At}x + \int_0^t e^{-A(t-s)}(-B(\omega s)\psi(s,x,\omega), \psi(s,x,\omega)) + f(\omega s))\mathrm{d}s.$$

There exist $\delta_1 = \delta_1(x_0,r) > 0$ and $T_1 = T_1(x_0,r) > 0$ such that $\Phi C_{x_0,r,\delta,T} \subseteq C_{x_0,r,\delta,T}$ for all $\delta \in (0,\delta_1]$ and $T \in (0,T_1]$. In fact,

$$\begin{aligned}
&|(\Phi\psi)(t,x,\omega) - x_0|_E \leqslant |e^{-At}x - x_0|_E \\
&\quad + \left| \int_0^t e^{-A(t-s)} B(\omega s)(\psi(s,x,\omega), \psi(s,x,\omega)))\mathrm{d}s \right|_E \\
&\quad + \left| \int_0^t e^{-A(t-s)} f(\omega s)\mathrm{d}s \right|_E \\
&\quad \leqslant m(\delta,T) + \int_0^t K e^{-a(t-s)}(t-s)^{-\alpha_1} |\psi(s,x,\omega)|_E^2 \, \mathrm{d}s \\
&\quad + \int_0^t K e^{-a(t-s)}(t-s)^{-\beta_1} \|f\| \mathrm{d}s \leqslant m(\delta,T) + K(|x_0|_E + r)^2 \frac{T^{1-\alpha_1}}{1-\alpha_1} \\
&\quad + K\|f\| \frac{T^{1-\beta_1}}{1-\beta_1} := d_1(x_0,r,\delta,T) \to 0
\end{aligned}$$

as $\delta + T \to 0$, where $m(\delta,T) := \sup\{|e^{-tA}x - x_0|_E : t \in [0,T], x \in B[x_0,r])$ and $\|f\| := \sup\{|f(\omega)|_X : \omega \in \Omega\}$. Thus there exist $\delta_1 = \delta_1(x_0,r) > 0$ and $T_1 = T_1(x_0,r) > 0$ such that $d_1(x_0,r,\delta,T) \leqslant r$ for all $\delta \in (0,\delta_1]$ and $T \in (0,T_1]$.

Let now $\psi_1, \psi_2 \in C_{x_0,r,\delta,T}$, then

$$\begin{gathered}
|(\Phi\psi_1)(t,x,\omega)) - (\Phi\psi_2)(t,x,\omega))|_E \\
= |\textstyle\int_0^t [B(\omega s)(\psi_1(s,x,\omega), \psi_1(s,x,\omega)) - B(\omega s)(\psi_2(s,x,\omega), \psi_2(s,x,\omega))]|_E \\
\leqslant 2C_B(|x_0|_E + r)Td(\psi_1,\psi_2)
\end{gathered}$$

and, consequently, $d(\Phi\psi_1, \Phi\psi_2) \leqslant L(x_0,r,T)d(\psi_1,\psi_2)$, where $L(x_0,r,T) = 2C_B(|x_0| + r)T \to 0$ as $T \to 0$. Thus there exists $T_2 = T_2(x_0,r) > 0$ such that $L(x_0,r,T) < 1$ for all $T \in (0,T_2]$. Denote by $\delta(x_0,r) := \delta_1(x_0,r)$ and $T(x_0,r) := min(T_1(x_0,r), T_2(x_0 r))$, then the mapping $\Phi : C_{x_0,r,\delta,T}$ is a contraction and, consequently, there exists a unique function $\varphi \in C_{x_0,r,\delta,T}$ satisfying the Eq. (22) on the interval $[0,T]$. The theorem is proved. □

Remark 2.5. Theorem 2.4 is true and for the equation

$$u' + Au = \mathcal{F}(\omega t, u)$$

if the continuous function $\mathcal{F}:\Omega \times E \to F$ satisfies the following conditions:

(i) $\sup\{|\mathcal{F}(\omega, 0)|_E : \omega \in \Omega\} < \infty$

(Ω, generally speaking, is not compact);

(ii) F is locally Lipschitz, i.e. for every $r > 0$ there exists $L(r) > 0$ such that

$$|\mathcal{F}(\omega, u_1) - \mathcal{F}(\omega, u_2)|_F \leqslant L(r)|u_1 - u_2|_E$$

for all $u_1, u_2 \in E$ with condition: $|u_i|_E \leqslant r (i = 1, 2)$.

Theorem 2.6. *Let $\mathcal{K}$ be a family of solutions of Eq. (22) satisfying the following condition: there exists a positive constant M such that $|x(t)|_{\mathcal{D}(A)} \leqslant M$ for all $t \in \mathbb{R}_+ (|x|_{\mathcal{D}(A)} := |Ax|_E)$. If there exists $\tilde{C}_B > 0$ such that*

$$|B(\omega)(u, v)|_F \leqslant \tilde{C}_B |u|_{\mathcal{D}(A)} |v|_{\mathcal{D}(A)}$$

for all $u, v \in \mathcal{D}(A)$, then this family of functions is uniform equicontinuous on $\mathbb{R}_+$, i.e., for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $|t_1 - t_2| < \delta$ implies $|x(t_1) - x(t_2)| < \varepsilon$ for all $t_1, t_2 \in \mathbb{R}_+$ and $x \in \mathcal{K}$.

Proof. Let $\psi \in \mathcal{K}$ and $x := \psi(0)$, then $\psi(t) = \varphi(t, x, \omega)$ for all $t \in \mathbb{R}_+$ and we have

$$\begin{aligned} |\varphi(t, x, \omega) - x|_E \leqslant\ & |e^{-At} x - x|_E \\ & + \Big| \int_0^t e^{-A(t-\tau)} (-B(\omega s)(\varphi(s, x, \omega), \varphi(s, x, \omega)) \\ & \quad + f(\omega s)) \mathrm{d}s \Big|_E \\ \leqslant\ & \int_0^t e^{-as} s^{-\alpha_1} |x|_{\mathcal{D}(A)} \mathrm{d}s \\ & + \int_0^t e^{-as} (t - s)^{-\alpha_1} C_b |\varphi(s, x, \omega)|^2_{\mathcal{D}(A)} \mathrm{d}s \\ & + \int_0^t e^{-a(t-s)} (t - s)^{-\beta_1} \|f\| \mathrm{d}s \\ \leqslant\ & \frac{t^{1-\alpha_1}}{1 - \alpha_1} M + C_B M^2 \frac{t^{1-\alpha_1}}{1 - \alpha_1} + \|f\| \frac{t^{1-\beta_1}}{1 - \beta_1}. \end{aligned} \tag{23}$$

From (23) we obtain

$$\sup\{|\varphi(t,x,\omega)-x|_E \ : \ |x|_{\mathcal{D}(A)}, \omega\in\Omega\}\to 0$$

as $t\to 0$ and, consequently,

$$\begin{aligned}|\varphi(t_2,x,\omega)-\varphi(t_1,x,\omega)|_E &= |\varphi(t_2-t_1,\varphi(t_1,x,\omega),\omega t_1)-\varphi(t_1,x,\omega)|_E\\ &\leqslant \sup\{|\varphi(t_2-t_1,x,\omega)-x|_E \ : \ |x|_{\mathcal{D}(A)}, \omega\in\Omega\}\to 0\end{aligned}$$

as $t_2-t_1\to 0$. The theorem is proved. □

Example 2.7. Navier–Stokes equations. We consider the two-dimensional Navier–Stokes system

$$\begin{aligned}&u'+q(t)\sum_{i=1}^{2}u_i\partial_i u=\nu\Delta u-\nabla p+\phi(t)\\ &\text{div } u=0, \quad u|_{\partial D}=0,\end{aligned} \tag{24}$$

where D is an open bounded set with smooth boundary $\partial D\in C^2$.

The functional setting of the problem is well known [20] and [28]. We denote by H and V the closures of the linear space $\{u\in C_0^\infty(D)^2, \text{div } u=0\}$ in $L^2(D)^2$ and $H_0'(D)^2$, respectively. Denote by P the corresponding orthogonal projection $P: L_2(D)^2\to H$. We further set

$$A:=-\nu P\Delta,\ B(t)(u,v):=q(t)P\left(\sum_{i=1}^{2}u_i\partial_i v\right).$$

The Stokes operator A is self-adjoint positive with domain $\mathcal{D}(A)$ dense in H. The inverse operator is compact. We define the Hilbert spaces $\mathcal{D}(A^\alpha)$, $\alpha\in(0,1]$ as the domains of the powers of A in the standard way. Furthermore, $V:=\mathcal{D}(A^{1/2})$, and $|u|_{\mathcal{D}(A^{1/2})}=|\nabla u|$.

Applying P we write (24) as the evolution equation of the following form

$$u'+Au+\mathcal{B}(t)(u,u)=\mathcal{F}(t),\ \mathcal{F}(t):=P\phi(t). \tag{25}$$

We suppose that $\mathcal{F}\in C(\mathbb{R},H)(X=H)$ and $\mathcal{B}\in C(\mathbb{R},L^2(H,\mathcal{D})(A^{-\delta}))$ $(F=\mathcal{D}(A^{-\delta}))$. Denote by $Y:=C(\mathbb{R},H)\times C(\mathbb{R},L^2(H,\mathcal{D}(A^{-\delta}))$ and $(Y,\mathbb{R},\sigma)$ a dynamical system of translations (Bebutov's dynamical system, see for example, [23–25]). Let $\Omega:=H(\mathcal{B},\mathcal{F})=\overline{\{(\mathcal{B}_\tau,\mathcal{F}_\tau)|\tau\in\mathbb{R}\}}$, where $\mathbb{B}_\tau(t):=\mathcal{B}(t+\tau)$ (respectively, $\mathcal{F}_\tau,(t):=\mathcal{F}(t+\tau)$) for all $t\in\mathbb{R}$, by bar we denote a closure in the compact-open topology and $(\Omega,\mathbb{R},\sigma)$ be a dynamical system of translations on Ω.

Along with Eq. (25) we consider its H-class

$$u' + Au + \tilde{\mathcal{B}}(t)(u,u) = \tilde{\mathcal{F}}(t), \tag{26}$$

where $(\tilde{\mathcal{B}}, \tilde{\mathcal{F}}) \in H(\mathcal{B}, \mathcal{F})$. Let $B : \Omega \to L^2(H, \mathcal{D})(A^{-\delta}))$ (respectively, $f : \Omega \to H$) be a mapping defined by equality

$$B(\omega) = B(\tilde{\mathcal{B}}, \tilde{\mathcal{F}}) := \tilde{\mathcal{B}}(0)(f(\omega) = f(\tilde{\mathcal{B}}, \tilde{\mathcal{F}}) := \tilde{\mathcal{F}}(0)),$$

where $\omega = (\tilde{\mathcal{B}}, \tilde{\mathcal{F}}) \in \Omega$, then Eq. (25) and its H-class can be written in the form (21).

We now set in the notation above $E = \mathcal{D}(A^{1/2})$, $X = H$, $F = \mathcal{D}(A^{-\delta})$ and see that (10)–(12), (18) and (16), (17) are valid with $\alpha_1 = 1/2 + \delta$, $\beta_1 = 1/2$, $\beta_2 = 3/2$.

We note that from the conditions (19)–(21) it follows that

$$|B(\omega)(x_1, x_1) - B(\omega)(x_2, x_2)|_F \leqslant C_B(|x_1|_E + |x_2|_E)|x_1 - x_2|_E \tag{27}$$

for all $x_1, x_2 \in E$ and $\omega \in \Omega$.

According to Theorem 2.4 through every point $x \in H$ passes a unique solution $\varphi(t, x, \omega)$ of Eq. (8) at the initial moment $t = 0$. And this solution is defined on some interval $[0, t_{(x,\omega)}]$. Let us note, that

$$\begin{aligned} w'(t) &= 2Re\langle \phi'(t,x,\omega), \varphi(t,x,\omega)\rangle = 2Re\langle A(\omega t)\varphi(t,x,\omega), \varphi(t,x,\omega)\rangle \\ &\quad + 2Re\langle B(\omega t)(\varphi(t,x,\omega), \varphi(t,x,\omega)), \varphi(t,x,y)\rangle + 2Re\langle f(\omega t), \varphi(t,x,\omega)\rangle \\ &= 2Re\langle A(\omega t)\varphi(t,x,\omega), \varphi(t,x,\omega)\rangle + 2Re\langle f(\omega t), \varphi(t,x,\omega)\rangle \\ &\leqslant -2\alpha|\varphi(t,x,\omega)|_E^2 + 2\|f\||\varphi(t,x,\omega)|_E, \end{aligned} \tag{28}$$

where $\|f\| := \max\{|f(\omega)|_X : \omega \in \Omega\}$ and $w(t) = |\varphi(t,x,\omega)|_E^2$. Then

$$w' \leqslant -2\alpha w + 2\|f\| w^{\frac{1}{2}} \tag{29}$$

and consequently

$$w(t) \leqslant v(t) \tag{30}$$

for all $t \in [0, t_{(x,\omega)})$, where $v(t)$ is an upper solution of equation

$$v' = -2\alpha v + 2\|f\| v^{\frac{1}{2}}, \tag{31}$$

satisfying condition $v(0) = w(0) = |x|^2$. Thus

$$v(t) = \left[\left(|x|_E - \frac{\|f\|}{\alpha}\right)e^{-\alpha t} + \frac{\|f\|}{\alpha}\right]^2 \tag{32}$$

and consequently

$$|\varphi(t,x,\omega)|_E \leqslant \left(|x|_E - \frac{\|f\|}{\alpha}\right) e^{-\alpha t} + \frac{\|f\|}{\alpha} \tag{33}$$

for all $t \in [0, t_{(x,\omega)})$. It follows from the inequality (29) that solution $\varphi(t,x,\omega)$ is bounded and therefore it may be prolonged on $\mathbb{R}_+ = [0, +\infty)$. Thus we have proved the following theorem.

Theorem 2.8. *Let the conditions (18) and (19) are fulfilled. Then the following statements hold:*

(i) *Every solution $\varphi(t,x,\omega)$ of non-autonomous Navier–Stokes equation (8) is bounded and therefore it may be prolonged on $\mathbb{R}_+$.*

(ii)
$$|\varphi(t,x,\omega)|_E \leqslant C(|x|_E), \tag{34}$$

for all $t \geqslant 0$, $\omega \in \Omega$ and $x \in E$, where $C(r) = r$ if $r \geqslant r_o := \frac{\|f\|}{\alpha}$ and $C(r) = r_o$ if $r \leqslant r_0$;

(iii)
$$\limsup_{t \to +\infty} \sup\{|\varphi(t,x,\omega)|_E \ : \ |x|_E \leqslant r, \omega \in \Omega\} \leqslant \frac{\|f\|}{\alpha} \tag{35}$$
for every $r > 0$.

Lemma 2.9. *Under the conditions of Theorem 2.8 we have*

$$\int_t^{t+l} |\varphi(\tau,x,\omega)|_E^2 \, \mathrm{d}\tau \leqslant \frac{r^2}{2\alpha} + \frac{r}{\alpha} l \|f\| := M(r) \tag{36}$$

for all $t \geqslant 0$ and $r \geqslant r_0$.

Proof. From the equality (29) after integration in t between t and $t+l$ we obtain

$$2\alpha \int_t^{t+l} |\varphi(t,x,\omega)|_E^2 \, \mathrm{d}\tau \leqslant |\varphi(\tau,x,\omega)|_E^2 + 2rl\|f\| \tag{37}$$

and, consequently,

$$\int_t^{t+l} |\varphi(\tau,x,\omega)|_E^2 \, \mathrm{d}\tau \leqslant \frac{r^2}{2\alpha} + \frac{r}{\alpha} l \|f\| := M(r). \tag{38}$$

□

Lemma 2.10. ***[29, Ch.3] (The Uniform Gronwall Lemma).*** *Let g, h, y, be three positive locally integrable functions on $]t_0, \infty[$ such that y' is locally integrable on $]t_0, \infty[$, and which satisfy*

$$y' \leqslant gy + h \quad \textit{for } t \geqslant t_0,$$

$$\int_t^{t+1} g(s)\mathrm{d}s \leqslant a_1, \quad \int_t^{t+1} h(s)\mathrm{d}s \leqslant a_2, \quad \int_t^{t+1} y(s)\mathrm{d}s \leqslant a_3 \quad \textit{for } t \geqslant t_0,$$

where l, a_l, a_2, a_3, are positive constants. Then

$$y(t+l) \leqslant \left(\frac{a_3}{l} + a_2\right) e^{a_1} \quad \forall t \geqslant t_0.$$

Theorem 2.11. *Under the conditions of Theorem 2.8 if*

$$|\langle B(\omega)(u,v), w\rangle| \leqslant C|u|^{1/2}|Au|^{1/2}|v|_{1/2}|w| \tag{39}$$

$$\forall u \in \mathcal{D}(A), v \in D(A^{1/2}), \quad w \in X, \tag{40}$$

then

$$|\varphi(t,x,\omega)|_{\mathcal{D}(A)} \leqslant K(r) \quad \forall |x| \leqslant r \, (r \geqslant r_0), \tag{41}$$

where $K(r)$ is some positive constant depending only on r.

Proof. Since

$$\langle A\varphi(t,x,\omega), \varphi(t,x,\omega)\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi(t,x,\omega)|_E^2 \tag{42}$$

by taking the scalar product of (8) with Au we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\varphi(t,x,\omega)|_E^2 + |A\varphi(t,x,\omega)|_E^2 + \langle B(\varphi(t,x,\omega),\varphi(t,x,\omega)), A\varphi(t,x,\omega)\rangle = \langle f(\omega t), A\varphi(t,x,\omega)\rangle. \tag{43}$$

Taking into account the inequality

$$|\langle f(\omega t), A\varphi(t,x,\omega)\rangle| \leqslant |f(\omega t)|_E |A\varphi(t,x,\omega)|_E \leqslant \frac{1}{4}|A\varphi(t,x,\omega)|^2 + \|f\|^2 \tag{44}$$

and using (42) and the Young inequality we obtain

$$\begin{aligned} &|\langle B(\omega)(\varphi(t,x,\omega),\varphi(t,x,\omega)), A\varphi(t,x,\omega)\rangle| \\ &\leqslant c_1|\varphi(t,x,\omega)|^{1/2}\|\varphi(t,x,\omega)\||A\varphi(t,x,\omega)|^{3/2} \\ &\leqslant \tfrac{1}{4}|A\varphi(t,x,\omega)|^2 + c_1'|\varphi(t,x,\omega)|^2\|\varphi(t,x,\omega)\|^4. \end{aligned} \tag{45}$$

Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t,x,\omega)\|^2 + |A\varphi(t,x,\omega)|^2 \leqslant \|f\|^2 + \frac{1}{2}|A\varphi(t,x,\omega)|^2 + c_1'|\varphi(t,x,\omega)|^2\|\varphi(t,x,\omega)\|^4. \tag{46}$$

From this inequality according to Gronwal lemma we can prove that $|\varphi(t,x,\omega)|_{\mathcal{D}(A)}$ is uniformly (w.r.t. x and ω) bounded on interval $[0,l]$. Applying the uniform Gronwal lemma with g, h, y replaced by.

$$c_1'|\varphi(t,x,\omega)|^2\|\varphi(t,x,\omega)\|^2, \quad \|f\|^2, \quad \|\varphi(t,x,\omega)\|^2 \tag{47}$$

we obtain that $\|\varphi(t,x,\omega)\|^2$ is bounded on $[l,\infty[$ and, consequently, it is bounded on $[0,\ \infty[$ uniformly w.r.t. $\|x\|\leqslant r$ and $\omega\in\Omega$. The theorem is proved. □

3. NON-AUTONOMOUS DISSIPATIVE DYNAMICAL SYSTEMS AND THEIR ATTRACTORS

Let Ω and W be two metric spaces, $(\Omega,\mathbb{R},\sigma)$ be a dynamical system on Ω. Let us consider a continuous mapping $\varphi:\mathbb{R}^+\times\Omega\times W$ satisfying the following conditions:

$$\varphi(0,\cdot,\omega)=id_W \quad \varphi(t+\tau,x,\omega)=\varphi(t,\varphi(\tau,x,\omega),\omega\tau)$$

for all $t,\tau\in\mathbb{R}^+$, $\omega\in\Omega$ and $x\in W$. Such mapping φ (or more explicit $\langle W,\varphi,(\Omega,\mathbb{R},\sigma)\rangle$) is called [1,6,25] a cocycle on $(\Omega,\mathbb{R},\sigma)$ with fiber W.

Remark 3.1. The non-autonomous Navier–Stokes equation (8) generates a cocycle φ (or more explicitly $\langle E,\varphi,(\Omega,\mathbb{R},\sigma)\rangle$), where $\varphi(t,x,\omega)$ is a unique solution of equation (8) defined on $\mathbb{R}_+$ with the initial condition $\varphi(0,x,\omega)=x$.

In fact, according to Theorem 2.4 the mapping $\varphi:\ \times E\times\Omega\to E((t,x,\omega)\to\varphi(t,x,\omega))$ is continuous and in view of uniqueness of solution $\varphi(t,x,\omega)$ we have the following identity: $\varphi(t+\tau,x,\omega)=\varphi(t,\varphi(\tau,x,\omega),\omega\tau)$ for all $t,\tau\in\mathbb{R}_+, x\in E$ and $\omega\in\Omega$, where $\omega\tau:=\sigma(\tau,\omega)$.

Example 3.2. Let E be a Banach space and $C(\mathbb{R}\times E,E)$ be a space of all continuous functions $F:\mathbb{R}\times E\to E$ equipped by the compact-open topology. Let us consider a parameterized differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}+Ax=F(\sigma_t\omega,x)(\omega\in\Omega)$$

on a Banach space E with $\Omega=C(\mathbb{R}\times E,E)$, where $\sigma_t\omega:=\sigma(t,\omega)$ and the linear operator A is densely defined in E and such that the linear equation

$$u'+Au=0$$

generates the c_0-semigroup of linear bounded operators

$$e^{-At}:E\to E,\varphi(t,x):=e^{-At}x.$$

We will define $\sigma_t : \Omega \to \Omega$ by $\sigma_t\omega(\cdot,\cdot) = \omega(t+\cdot,\cdot)$ for each $t \in \mathbb{R}$ and interpret $\varphi(t,x,\omega)$ as mild solution of the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) + Ax = F(\sigma_t\omega, x(t)), \quad x(0) = x. \tag{48}$$

Under appropriate assumptions on $F : \Omega \times E \to E$ (or even $F : \mathbb{R} \times E \to E$ with $\omega(t)$ instead of $\sigma_t\omega$ in (48)) to ensure forwards existence and uniqueness, then φ is a cocycle on $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ with fiber E, where $(C(\mathbb{R} \times E, E), \mathbb{R}, \sigma)$ is a Bebutov's dynamical system (see for example [3,7, 23,25]).

The triplet $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h \rangle$, where $h : X \to \Omega$ is a homomorphism from the dynamical system $(X, \mathbb{R}_+, \pi)$ onto $(\Omega, \mathbb{R}, \sigma)$, is called (see [4,7]) a non-autonomous dynamical system.

Let φ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with the fiber E. Then the mapping $\pi : \mathbb{R}^+ \times \Omega \times E \to \Omega \times E$ defined by

$$\pi(t,x,\omega) := (\varphi(t,x,\omega), \sigma_t\omega)$$

for all $t \in \mathbb{R}^+$ and $(x,\omega) \in E \times \Omega$ forms a semi-group $\{\pi(t,\cdot,\cdot)\}_{t \in \mathbb{R}^+}$ of mappings of $X := \Omega \times E$ into itself, thus a semi-dynamical system on the state space X, which is called a skew-product flow [25] and the triplet $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h \rangle$ (where $h := pr_2 : X \to \Omega$) is a non-autonomous dynamical system.

The cocycle φ over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W we will define by a compact (bounded) dissipative one, if there is a nonempty compact $K \subseteq W$ such that

$$\lim_{t \to +\infty} \sup\{\beta(U(t,\omega)M, K) | \omega \in \Omega\} = 0 \tag{49}$$

for any $M \in C(W)$ (respectively $M \in \mathcal{B}(W)$), where by $C(W)(\mathcal{B}(W))$ is denoted a family of all compact (bounded) subsets of W, by β a semi-distance of Hausdorff and $U(t,\omega) := \varphi(t,\cdot,\omega)$.

Lemma 3.3. *Let Ω be a compact metric space and $\langle (W, \varphi, (\Omega, \mathbb{R}, \sigma) \rangle$ be a cocycle over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W. In order to $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma) \rangle$ be a compact (bounded) dissipative, it is necessary and sufficient that the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ should be a compact (bounded) dissipative one.*

This assertion directly follows from the corresponding definitions (see for example [7,14]).

By the whole trajectory of the semi-group dynamical system $(X, \mathbb{R}_+, \pi)$ (of the cocycle $\langle (W, \varphi, (\Omega, \mathbb{R}, \sigma) \rangle$ over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W), which passes through the point $x \in X((u, y) \in W \times \Omega)$ we will call the continuous mapping $\gamma: \mathbb{R} \to X(\nu: \mathbb{R} \to W)$ which satisfies the conditions: $\gamma(0) = x(\nu(0) = u)$ and $\pi^t \gamma(\tau) = \gamma(t+\tau)(\nu(t+\tau) = \varphi(t, \nu(\tau), \omega t))$ for all $t \in \mathbb{R}_+$ and $\tau \in \mathbb{R}$. If $M \subseteq W$, then we denote by

$$\Omega_\omega(M) = \bigcap_{t \geqslant 0} \overline{\bigcup_{\tau \geqslant t} \varphi(\tau, M, \omega^{-\tau})}$$

for every $\omega \in \Omega$, where $\omega^{-\tau} := \sigma(-\tau, \omega)$.

Theorem 3.4. *[5,7]. Let Ω be a compact metric space, $\langle (W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ be a compactly (boundedly) dissipative cocycle and K be a nonempty compact, arising in the equality (49), then the following assertions hold:*

(i) *$I_\omega := \Omega_\omega, (K) \neq \emptyset$, is compact, $I_\omega \subseteq K$ and*

$$\lim_{t \to +\infty} \beta(U(t, \omega^{-t})K, I_\omega) = 0$$

for every $\omega \in \Omega$;

(ii) *$U(t, \omega) I_\omega = I_{\omega t}$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$;*

(iii)

$$\lim_{t \to +\infty} \beta(U(t, \omega^{-t})M, I_\omega) = 0$$

for all $M \in C(W)$ (respectively, $M \in \mathcal{B}(X)$) and $\omega \in \Omega$;

(iv)

$$\lim_{t \to +\infty} \sup\{\beta(U(t, \omega^{-t})M, I)|_\omega \in \Omega\} = 0$$

for any $M \in C(W)$ (respectively, $M \in \mathcal{B}(X)$), where $I = \cup\{I_\omega | \omega \in \Omega\}$;

(v) *$I_\omega := pr_1 J_\omega$ for all $\omega \in \Omega$, where J is a Levinson's center of $(X, \mathbb{R}_+, \pi)$, and, hence, $I = pr_1 J$;*

(vi) *the set I is compact;*

(vii) *the set I is connected if the spaces W and Y are connected.*

The family of compact sets $\{I_\omega | \omega \in \Omega\}$ ($I_\omega \subset W$ is nonempty compact for every $\omega \in \Omega$) is called (see, for example, [5] or [7]) the compact global attractor of cocycle φ if the following conditions are fulfilled:

(i) The set $I := \bigcup\{I_\omega | \omega \in \Omega\}$ is precompact.

(ii) $\{I_\omega | \omega \in \Omega\}$ is invariant w.r.t. the cocycle φ, i.e. $\varphi(t, \omega, I_\omega) = I_{\sigma_t \omega}$, for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

(iii) The equality $\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, K, \omega), I) = 0$ holds for every bounded set $K \subset W$.

Corollary 3.5. *Under the conditions of Theorem 3.4 the cocycle φ admits a compact global attractor.*

Dynamical system $(X, \mathbb{R}_+, \pi)$ is called asymptotically compact (see [14, 19] and also [5, 7]) if for any positive invariant bounded set $A \subset X$ there is a compact $K_A \subset X$ such that

$$\lim_{t\to+\infty} \beta(\pi^t A, K_A) = 0. \tag{50}$$

Dynamical system $(X, \mathbb{R}_+, \pi)$ is called compact (completely continuous) if for every bounded set $A \subset X$ there exists a positive number $l = l(A)$ such that the set $\pi^l A$ will be precompact.
It is easy to verify (see for example [7]) that every compact dynamical system $(X, \mathbb{R}_+, \pi)$ is asymptotically compact.

The cocycle $\langle (W, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is called compact (asymptotically compact) if the skew-product dynamical system $(X, \mathbb{R}_+, \pi)(X = W \times Y, \pi = (\varphi, \sigma))$ is compact (respectively asymptotic compact).
Let $(X, \mathbb{R}_+, \pi)$ be compact dissipative and K be a compact set, which attracts all compact subsets of X. Suppose

$$J = \Omega(K), \tag{51}$$

where $\Omega(K) = \bigcap_{t\geqslant 0} \overline{\bigcup_{\tau\geqslant t} \pi^\tau K}$. The set J defined by the equality (51) does not depend on selection of the attractor K, and is characterized only by the properties of the dynamical system $(X, \mathbb{R}_+, \pi)$ itself. The set J is called the Levinson's center of the compact dissipative system $(X, \mathbb{R}_+, \pi)$.

Theorem 3.6. *[5, 7]. Let (E, Ω, h) be a local-trivial Banach fibering, $\langle (E, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ be a non-autonomous dynamical system and the dynamical system $\langle (E, \mathbb{R}_+, \pi)$ be completely continuous. Then next conditions are equivalent:*

(i) *there is a positive number r such that for any $x \in X$ there will be $\tau = \tau(x) \geqslant 0$ for which $|x\tau| < r$;*
(ii) *the dynamical system $(E, \mathbb{R}_+, \pi)$ is compact dissipative and*

$$\lim_{t\to+\infty} \sup_{|x|\leqslant R} \rho(xt, J) = 0$$

for any $R > 0$, where J is a Levinson's center of dynamical system $(E, \mathbb{R}_+, \pi)$, that is the non-autonomous system $\langle (E, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ admits a compact global attractor J.

A dynamical system $(X, \mathbb{R}_+, \pi)$ satisfies the condition of Ladyzhenskaya (see [19] and also [7]) if for any bounded set $A \subset X$ there is a compact $K_A \subset X$ such that the equality (50) holds.

Theorem 3.7. *[5,7]. Let $\langle (E, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ be a non-autonomous dynamical system and let $(E, \mathbb{R}_+, \pi)$ satisfy the condition of Ladyzhenskaya. Then the conditions 1. and 2. of Theorem 3.6 are equivalent.*

Applying general theorems about non-autonomous dissipative systems to non-autonomous system constructed in Example 3.2, we will obtain series of facts concerning Eq. (8). In particular, from Theorems 2.8, 3.4 and 3.7 follows the theorem below.

Theorem 3.8. *Let Ω be a compact metric space, $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω and the conditions (18) and (19) are fulfilled. If the cocycle φ generated by non-autonomous Navier–Stokes equation is asymptotically compact, then for every $\omega \in \Omega$ there exists a non-empty compact and connected $I_\omega, \subset E$ such that the following conditions hold:*

(i) *the set $I = \cup\{I_\omega : \omega \in \Omega\}$ is compact in E;*
(ii) *I is connected if Ω is connected;*
(iii)

$$\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(U(t, \omega^{-t})M, I) = 0$$

for any bounded set $M \subset E$, where $U(t, \omega) = \varphi(t, \cdot, \omega)$ and β is the semi-distance of Hausdorff;
(iv) *$U(t, \omega)I_\omega = I_{\omega t}$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;*
(v) *I_ω consists of those and only those points $x \in E$ through which passes the solution of Eq. (8) bounded on $\mathbb{R}$.*

Theorem 3.9. *Under conditions of Theorem 3.8*

$$|\varphi(t, x, \omega)| \leqslant \frac{\|f\|}{\alpha}$$

for all $t \in \mathbb{R}$, $\omega \in \Omega$ and $x \in I_\omega$, where φ is a cocycle, generated by non-autonomous Navier–Stokes equation.

Proof. According to Theorem 3.4 the set $J = \bigcup\{I_\omega \times \{\omega\} : \omega \in \Omega\}$ is a Levinson's center of dynamical system $(X, \mathbb{R}_+, \pi)$ and according to (51) for any point $(u_0, y_0) = z \in J$ there exists $t_n \to +\infty, u_n \in E$ and $\omega_n \in \Omega$ such that the sequence $\{u_n\}$ is bounded, $u_0 = \lim_{n\to+\infty} \varphi(t_n, u_n, \omega_n)$ and $\omega_0 \in \lim_{n\to+\infty} \omega_n t_n$. From the inequality (29) follows that $|u_0| \leqslant \|f\|/\alpha$, i.e.,

$\varphi(t,x,\omega)\in I_{\omega t}$ for all $\omega\in\Omega$ and $t\geqslant 0$, hence $|\varphi(t,x,\omega)|\leqslant\|f\|/\alpha$ for any $t\in\mathbb{R}, x\in I_\omega$, and $\omega\in\Omega$. The theorem is proved. □

4. ALMOST PERIODIC AND RECURRENT SOLUTIONS OF NON-AUTONOMOUS NAVIER–STOKES EQUATIONS

Let $\mathbb{T}=\mathbb{R}$ or $\mathbb{R}_+$ and $(X,\mathbb{T},\pi)$ be a dynamical system.

Definition 4.1. The point $x\in X$ is called a stationary (τ-periodic, $\tau>0, \tau\in\mathbb{T}$) point, if $xt=x$ ($x\tau=x$ respectively) for all $t\in\mathbb{T}$, where $xt:=\pi(t,x)$.

Definition 4.2. The number $\tau\in\mathbb{T}$ is called $\varepsilon>0$ shift (almost period) of point $x\in X$ if $\rho(x\tau,x)<\varepsilon$ (respectively $\rho(x(\tau+t),xt)<\varepsilon$ for all $t\in\mathbb{T}$).

Definition 4.3. The point $x\in X$ is called almost recurrent (almost periodic) if for any ε there exists a positive number l such that on any segment of length l, will be found a ε shift (almost period) of point $x\in X$.

Definition 4.4. If a point $x\in X$ is almost recurrent and the set $H(x)=\overline{\{xt|t\in\mathbb{T}\}}$ is compact, then x is called recurrent.

Definition 4.5. An autonomous dynamical system $(\Omega,\mathbb{T},\sigma)$ is said to be pseudo recurrent if the following conditions are fulfilled:

(a) Ω *is compact;*
(b) $(\Omega,\mathbb{T},\sigma)$ *is transitive, i.e., there exists a point* $\omega_0\in\Omega$ *such that* $\Omega=\overline{\{\omega_0 t|t\in\mathbb{T}\}}$*;*
(c) *every point* $\omega\in\Omega$ *is stable in the sense of Poisson, i.e.*

$$\mathfrak{N}_\omega=\{\{t_n\}|\omega t_n\to\omega \text{ and } |t_n|\to+\infty\}\neq\emptyset.$$

Definition 4.6. A point $x\in X$ is said to be pseudo-recurrent is the dynamical system $(H(x),\mathbb{T},\pi)$ is pseudo-recurrent.

Lemma 4.7. *[10]. Let* $\langle(X,\mathbb{T}_1,\pi),(\Omega,\mathbb{T}_2,\sigma),h\rangle$ *be a non-autonomous dynamical system and the following conditions are fulfilled:*

(1) $(\Omega,\mathbb{T}_2,\sigma)$ *is pseudo-recurrent;*
(2) $\gamma\in C(\Omega,X)$ *is an invariant section of the homomorphism* $h:X\to\Omega$*, i.e.* $h(\gamma(\omega))=\omega$ *and* $\gamma(\sigma(t,\omega))=\pi(t,\gamma(\omega))$ *for all* $\omega\in\Omega$ *and* $t\in\mathbb{T}_2$.

Then the autonomous dynamical system $(\gamma(\Omega),\mathbb{T}_2,\pi)$ *is pseudo-recurrent.*

Let $\mathbb{T}=\mathbb{S}$ and $(X,\mathbb{S},\pi)$ be a bi-sided dynamical system.

Definition 4.8. A recurrent point $x \in X$ is called almost automorphic (see, for example [27]) if whenever t_α is a net with $xt_\alpha \to x_*$, then $x_*(-t_\alpha) \to x$ too.

Definition 4.9. A motion $\varphi(t, u_0, y_0)$ ($u_0 \in E$ and $y_0 \in Y$) of the cocycle φ is called recurrent (almost periodic, almost automorphic, quasi-periodic, periodic), if the point $x_0 := (u_0, y_0) \in X := E \times Y$ is a recurrent (almost periodic, almost automorphic, quasi-periodic, periodic) point of the skew-product dynamical system $(X, \mathbb{S}_+, \pi)$ $(\pi := (\varphi, \sigma))$.

Remark 4.10. We note (see, for example, [21–24]) that if $y \in Y$ is a stationary (respectively τ-periodic, almost periodic, quasi periodic, recurrent) point of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ and $h : Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ onto $(X, \mathbb{T}_1, \pi)$, then the point $x = h(y)$ is a stationary (respectively τ-periodic, almost periodic, quasi periodic, recurrent) point of the system $(X, \mathbb{T}_1, \pi)$.

Lemma 4.11. *If $y \in Y$ is an almost automorphic point of the dynamical system $(Y, \mathbb{T}, \sigma)$ and $h : Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{S}, \sigma)$ onto $(X, \mathbb{S}_+, \pi)$, then the point $x = h(y)$ is an almost automorphic point of the system $(X, \mathbb{S}_+, \pi)$.*

Proof. Let t_α be a net with $xt_\alpha \to x_*$, then we have $yt_\alpha \to y_*$ ($y := h(x)$ and $y_* := h(x_*)$). Since the point y is almost automorphic, then also $y_*(-t_\alpha) \to y$ and, consequently, $x_*(-t_\alpha) = h(y_*(-t_\alpha)) \to h(y) = x$. The lemma is proved. □

Remark 4.12. Let $X := E \times Y$ and $\pi := (\varphi, \sigma)$. Then mapping $h: Y \to X$ is a homomorphism of the dynamical system $(Y, \mathbb{T}_2, \sigma)$ onto $(X, \mathbb{T}_1, \pi)$ if and only if $h(y) = (\gamma(y), y)$ for all $y \in Y$, where $\gamma: Y \to E$ is a continuous mapping with the condition that $\gamma(yt) = \varphi(t, \gamma(y), y)$ for all $y \in Y$ and $t \in \mathbb{T}_2$.

Definition 4.13. The solution $\varphi(t, x, \omega)$ of non-autonomous Navier–Stokes equation (8) is called recurrent (respectively pseudo-recurrent, almost automorphic, almost periodic, quasi-periodic), if the point $(x, \omega) \in H \times \Omega$ is a recurrent (respectively pseudo-recurrent, almost automorphic, almost periodic, quasi-periodic) point of skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ $(X = H \times \Omega$ and $\pi = (\varphi, \sigma))$.

Let $X = H \times \Omega$ and $\pi = (\varphi, \sigma)$, then mapping h: $\Omega \to X$ is a homomorphism of dynamical system $(\Omega, \mathbb{R}, \sigma)$ onto $(X, \mathbb{R}_+, \pi)$ if and only if $h(\omega) = (u(\omega), \omega)$ for all $\omega \in \Omega$, where u: $\Omega \to H$ is a continuous mapping with the condition that $u(\omega t) = \varphi(t, u, (\omega), \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$.

The following affirmations hold:

Lemma 4.14. *Let Ω be a compact metric space, A be a linear operator densely defined in E such that the equation*

$$x' + Ax = 0$$

generates a c_0-semigroup $\{U(t)\}_{t \geqslant 0}$. If the condition (18) is fulfilled, then

$$\|U(t)\| \leqslant \exp(-\alpha t)$$

for all $t \in \mathbb{R}_+$, where $U(t)$ is a Cauchy's operator of equation (52).

Proof. Let $\varphi(t,x) := U(t)x$, then according to the inequality 18 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|\varphi(t,x)|^2 \leqslant -2\alpha|\varphi(t,x)|^2$$

and consequently, $|\varphi(t,x)| \leqslant \exp(-\alpha t)|x|$ for all $x \in H$ and $t \in \mathbb{R}_+$. Thus we have $|U(t)x| \leqslant \exp(-\alpha t)|x|$, therefore $\|U(t)\| \leqslant \exp(-\alpha t)$ for all $t \in \mathbb{R}_+$.

Lemma 4.15. *Suppose that the condition (18) is fulfilled. Then for every function $f \in C(\Omega, H)$ there exists a unique function $\gamma \in C(\Omega, H)$ defined by equality*

$$\gamma(\omega) = \int_{-\infty}^{0} U(-\tau) f(\omega\tau)\ \mathrm{d}\tau$$

such that

$$\gamma(\omega t) = \varphi(t, \gamma(\omega), \omega) \tag{52}$$

for every $\omega \in \Omega$ and $t \in \mathbb{R}_+$, where $\varphi(t,x,\omega)$ is a solution of equation

$$u' = Au + f(\omega t)$$

with the initial condition $\varphi(0,x,\omega) = x$ and the following inequality

$$\|\gamma\| \leqslant \frac{1}{\alpha}\|f\|$$

takes place.

Proof. The formulated statement results from Lemma 4.14 and Proposition 7.33 from [12]. □

Lemma 4.16. *Let Ω be a compact metric space, the cocycle φ, generated by the non-autonomous Navier–Stokes equation (8) and $\alpha^{-2}\|f\|C_B<1$, then the following inequality*

$$|\varphi(t,x_1,\omega)-\varphi(t,x_2,\omega)|\leqslant e^{-(\alpha-C_B\frac{\|f\|}{\alpha})t}|x_1-x_2|$$
$$(x_1,x_2\in B[0,r_0],\omega\in\Omega \text{ and } t\in\mathbb{R}_+)$$

takes place.

Proof. Let $r_0:=\|f\|/\alpha$ and $x_1,x_2\in B[0,r_0]:=\{x\in E:|x|\leqslant r_0\}$. According to Theorem 2.8 we have $|\varphi(t,x_i,\omega)|\leqslant r_0$ for all $t\geqslant 0,\omega\in\Omega$ and $i=1,2$. Denote by $\psi(t):=\varphi(t,x_1,\omega)-\varphi(t,x_2,\omega)$, then we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)|^2 &=2Re\langle A\psi(t),\psi(t)\rangle+2Re\langle B(\omega t)(\psi(t),\varphi(t,x_2,\omega)),\psi(t)\rangle\\ &\leqslant -2\alpha|\psi(t)|^2+2C_B|\varphi(t,x_2,\omega)\||\psi(t)|^2\\ &\leqslant -2\alpha|\psi(t)|^2+2C_B\frac{\|f\|}{\alpha}|\psi(t)|^2=-2\Big(\alpha-C_B\frac{\|f\|}{\alpha}\Big)|\psi(t)|^2\end{aligned}$$

and, consequently,

$$|\psi(t)|^2\leqslant e^{-2\left(\alpha-C_B\frac{\|f\|}{\alpha}\right)t}|\psi(0)|^2.$$

Thus we have

$$|\varphi(t,x_1,\omega)-\varphi(t,x_2,\omega)|\leqslant e^{-\left(\alpha-C_B\frac{\|f\|}{\alpha}\right)t}|x_1-x_2|$$
$$(x_1,x_2\in B[0,r_0],\quad \omega\in\Omega\quad \text{and}\quad t\in\mathbb{R}_+)$$

for all $x_1,x_2\in B[0,r_0],\omega\in\Omega$ and $t\in\mathbb{R}_+$. The lemma is proved. □

Theorem 4.17. *Let $r_0:=\|f\|/\alpha$ and Ω be a compact metric space, the cocycle φ, generated by the non-autonomous Navier–Stokes equation (8) and $\|f\|C_B/\alpha^2<1$. Then there exists a function $\gamma\in C(\Omega,B[0,r_0])$ such that:*

$$\text{(a)}\qquad \gamma(\omega t)=\varphi(t,\gamma(\omega),\omega) \tag{53}$$

for every $\omega\in\Omega$ and $t\in\mathbb{R}_+$, where $\varphi(t,x,\varphi)$ is a solution of Eq. (8) with the initial condition $\varphi(0,x,\omega)=x$;

$$\text{(b)}\qquad \|\gamma\|\leqslant\frac{\|f\|}{\alpha}; \tag{54}$$

$$\text{(c)}\qquad |\varphi(t,x,\omega)-\gamma(\omega t)|\leqslant e^{-\left(\alpha-C_B\frac{\|f\|}{\alpha}\right)t}|x-\gamma(\omega)| \tag{55}$$

for all $x\in E,\omega\in\Omega$ and $t\in\mathbb{R}_+$, where $\|\gamma\|:=\sup\{|\gamma(\omega)|:\omega\in\Omega\}$.

Proof. Let $\Gamma := C(\Omega, B[0, r_0])(C(\Omega, E))$ be a space all the continuous functions $f\colon \Omega \to B[0, r_0]$ (respectively $f\colon \Omega \to E$) equipped with the distance

$$d(f_1, f_2) = \max\{|f_1(\omega) - f_2(\omega)| : \omega \in \Omega\}.$$

Then (Γ, d) (respectively $(C(\Omega, E), d)$) is a complete metric space.
Let $t \in \mathbb{R}_+$. We define the mapping $S^t\colon \Gamma \to C(\Omega, E)$ by the equality

$$(S^t \nu)(w) := U(t, \omega^{-t})\nu(\omega^{-t})$$

for all $w \in \Omega$, where $\omega^{-t} := \sigma(-t, \omega)$ and $U(t, \omega) := \varphi(t, \cdot, \omega)$. According to Theorem 2.8 we have $S^t(\Gamma) \subseteq \Gamma$ for all $t \in \mathbb{R}_+$. It easy to see that the family of mappings $\{S^t | t \in \mathbb{R}_+\}$ possesses the following properties:

$$\text{(i)} \qquad S^0 = Id_\Gamma$$

and

$$\text{(ii)} \qquad S^{t+\tau} = S^t S^\tau$$

for all $t, \tau \in \mathbb{R}_+$.

Thus $\{S^t | t \in \mathbb{R}_+\}$ forms a commutative semigroup with identity element. Now we will show that the mapping $S^t (t > 0)$ is a contraction. In fact, let $\nu_1, \nu_2 \in \Gamma$, then we have

$$(S^t \nu_1)(\omega) - (S^t \nu_2)(\omega) = U(t, \omega^{-t})\nu_1(\omega^{-t}) - U(t, \omega^{-t})\nu_2(\omega^{-t}). \qquad (56)$$

From Lemma 4.16 and the equality (56) it follows that

$$d(S^t \nu_1, S^t \nu_2) \leqslant e^{-\left(\alpha - C_B \frac{\|f\|}{\alpha}\right)t}\, d(\nu_1, \nu_2)$$

for all $t \in \mathbb{R}_+$ and, consequently, there exists a unique common fixe point $\gamma \in \Gamma$, i.e., $S^t\gamma = \gamma$ for all $t \in \mathbb{R}_+$. In particularly

$$U(t, \omega^{-t})\gamma(\omega^{-t}) = \gamma(\omega)$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$. From this equality follows that

$$\gamma(\omega t) = U(t, \omega)\gamma(\omega) = \varphi(t, \gamma(\omega), \omega)$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

Let $x \in E$, $\varphi(t,x,\omega)$ be a unique solution of Eq. (8) with the initial condition $\varphi(0,x,\omega)=x$ and $\gamma \in \Gamma$ the function with the property (53). Denote by $\psi(t):=\varphi(t,x,\omega)-\gamma(\omega t)$, then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)|^2 &= 2Re\langle A\psi(t),\psi(t)\rangle + 2Re\langle B(\omega t)(\psi(t),\gamma(\omega(t)),\psi(t)\rangle \\ &\leqslant -2\alpha|\psi(t)|^2 + 2C_B|\gamma(\omega t)||\psi(t)|^2 \leqslant -2\alpha|\psi(t)|^2 \\ &\quad +2C_B\frac{\|f\|}{\alpha}|\psi(t)|^2 = -2\left(\alpha - C_B\frac{\|f\|}{\alpha}\right)|\psi(t)|^2\end{aligned}$$

and, consequently,

$$|\psi(t)|^2 \leqslant e^{-2\left(\alpha - C_B\frac{\|f\|}{\alpha}\right)t}|\psi(0)|^2.$$

Thus we have

$$|\varphi(t,x,\omega)-\gamma(\omega t)| \leqslant e^{-\left(\alpha - C_B\frac{\|f\|}{\alpha}\right)t}|x-\gamma(\omega)|$$

for all $x \in E$, $\omega \in \Omega$ and $t \in \mathbb{R}_+$. The theorem is proved. □

Corollary 4.18. *Under the conditions of Theorem 4.17 there exists a unique function $\gamma \in C(\Omega, E)$ such that*

$$\gamma(\omega t) = \varphi(t, \gamma(\omega), \omega) \tag{57}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

Proof. Let $\bar{\gamma} \in C(\Omega, E)$ be a function satisfying the equality (57) and $\gamma E \Gamma = C(\Omega, B[0, r_0])$ the function from Theorem 4.17. Since $\bar{\gamma}(\omega t) = \varphi(t, \bar{\gamma}(\omega), \omega)$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, then according to the inequality (55) we have

$$|\bar{\gamma}(\omega t)-\gamma(\omega t)| \leqslant e^{-\left(\alpha - C_B\frac{\|f\|}{\alpha}\right)t}\bar{\gamma}|(\omega)-\gamma(\omega)| \tag{58}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$. In particularly, from (58) we obtain

$$\begin{aligned}|\bar{\gamma}(\omega)-\gamma(\omega)| &\leqslant e^{-\left(\alpha - C_B\frac{\|f\|}{\alpha}\right)t}|\bar{\gamma}(\omega^{-t})-\gamma(\omega^{-t})| \\ &\leqslant e^{-\left(\alpha - C_B\frac{\|f\|}{\alpha}\right)t}\|\bar{\gamma}-\gamma\|\end{aligned} \tag{59}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, where $\omega^{-t} := \sigma(-t, \omega)$ and $\|\bar{\gamma}-\gamma\| := \max\{\bar{\gamma}(\omega) - \gamma(\omega)| : \omega \in \Omega\}$. Passing to the limit in the inequality (59) we obtain $\bar{\gamma}(\omega) = \gamma(\omega)$ for all $\omega \in \Omega$. □

Corollary 4.19. *Under the conditions of Theorem 4.17, Eq. (8) admits a compact global attractor $\{I_\omega : \omega \in \Omega\}$ and $I_\omega = \{\gamma(\omega)\}$ for all $\omega \in \Omega$, where $\gamma \in \Gamma$ is a function from Theorem 4.17.*

Corollary 4.20. *The following statements hold:*

(i) *Let Ω be a compact minimal set containing only the periodic (respectively quasi periodic, almost periodic, almost automorphic, recurrent) motions, then under conditions of Theorem 4.17 the non-autonomous Navier–Stokes Eq. (8) admits a unique periodic (respectively, quasi-periodic, almost periodic, almost automorphic, recurrent) solution $\gamma(\omega t)$ and every other solution of this equation is asymptotically periodic (respectively asymptotically quasi-periodic, asymptotically almost periodic, asymptotically automorphic, asymptotically recurrent).*

(ii) *If $(\Omega, \mathbb{T}, \sigma)$ is a pseudo-recurrent dynamical system, then under conditions of Theorem 4.17 the non-autonomous Navier–Stokes equation (8) admits a unique pseudo-recurrent solution $\gamma(\omega t)$ and every other solution of this equation is asymptotically pseudo-recurrent.*

Proof. Let $\gamma \in \Gamma$ be a function from Theorem 4.17, then according this theorem we have $\varphi(t, \gamma(\omega), \omega) = \gamma(\omega t)$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$ and, consequently, the solution $\varphi(t, \gamma(\omega), \omega)$ is periodic (quasi periodic, almost periodic, almost automorphic, recurrent, pseudo-recurrent). Let $\varphi(t, x, \omega)$ be a arbitrary solution of Eq. (8), then taking into consideration the inequality (55) we conclude that $\varphi(t, x, \omega)$ is asymptotically periodic (asymptotically quasi-periodic, asymptotically almost periodic, asymptotically almost automorphic, asymptotically recurrent, asymptotically pseudo-recurrent). □

5. UNIFORM AVERAGING FOR A FINITE INTERVAL

We shall be dealing with the non-autonomous Navier–Stokes equation

$$u' + \varepsilon Au + \varepsilon B(u, u) = \varepsilon f(\omega t), \tag{60}$$

where $\varepsilon \in [0, \varepsilon_0]$, A is linear and B is a bilinear operator, f is a forcing term.

Below we will use some notions, denotations and results from [18]. Let Banach spaces $E, F, X, \mathcal{E}$ satisfy

$$E \subset F; \quad E, F, X \subset \mathcal{E},$$

each embedding being dense and continuous.

We suppose that the linear equation

$$u' = Au \tag{61}$$

generates the c_0-semigroup of linear bounded operators

$$e^{At}:\ \mathcal{E}\to\mathcal{E}, \tag{62}$$

which for $t>0$ can be extended to the linear bounded operators from F to E satisfying the estimates (10)–(12).

We also suppose that the following condition is satisfied

$$Ae^{At}=e^{At}A, \tag{63}$$

in the sense of $L(F,E) := \{A:F\to E | A \text{ is linear and bounded}\}$ equipped with the operational norm.

Function f. The external force $f:\Omega\to X$ is continuous, i.e. $f\in C(\Omega,X)$.

Operators e^{At}. The operators $e^{-At}(t>0)$ can be extended to the linear bounded operators from X to E satisfying the estimates (16)–(17) and Eq. (63), this time in the sense of (X,E).

Existence of partial averaged. $f(\omega)=f_0(\omega)+f_1(\omega)(f_0,f_1\in C(\Omega,X))$ for all $\omega\in\Omega$ and the average of $f_1(\omega)$ is equal to 0, that is,

$$\lim_{t\to\infty}\ \frac{1}{t}\int_0^t(\omega\tau)\mathrm{d}\tau=0, \tag{64}$$

uniformly with respect to $\omega\in\Omega$.

Remark 5.1. 1. The condition (64) is fulfilled if a dynamical system $(\Omega,\mathbb{R},\sigma)$ is strictly ergodic, i.e., on Ω exists a unique invariant measure μ w.r.t. $(\Omega,\mathbb{R},\sigma)$.
2. According to Lemma 5.1 from [9] the equality (64) takes place if and only if there exists a positive continuous on $\mathbb{R}_+$ function k with $\lim_{t\to\infty}k(t)=0$ such that

$$\left|\frac{1}{t}\int_0^t f_1(\omega\tau)\mathrm{d}\tau\right| x\leqslant k(t) \tag{65}$$

for all $\omega\in\Omega$ and $t\in\mathbb{R}_+$.

Along with Eq. (60) we consider also the partial averaged equation

$$u'+\varepsilon Ax+\varepsilon B(u,u)=\varepsilon f_0(\omega t). \tag{66}$$

If we introduce the "slow time" $\tau:=\varepsilon t(\varepsilon>0)$, then Eqs. (60) and (66) can be written in the following way

$$u'+Au+B(u,u)=f\left(\omega\frac{\tau}{\varepsilon}\right) \tag{67}$$

and

$$\bar{u}' + A\bar{u} + B(\bar{u}, \bar{u}) = f_0\left(\omega\frac{\tau}{\varepsilon}\right). \tag{68}$$

We will consider the mild solutions $u(t)$ and $\bar{u}(t)$ of Eqs. (67) and (68), i.e. $u, \bar{u} \in C([0, T], E)$ and satisfy the following integral equations

$$u(\tau) = e^{-A\tau}x + \int_0^\tau e^{-A(\tau-s)}\left(-B(u(s), u(s)) + f\left(\omega\frac{s}{\varepsilon}\right)\right)\mathrm{d}s, \tag{69}$$

and

$$\bar{u}(\tau) = e^{-A\tau}x + \int_0^\tau e^{-A(\tau-s)}\left(-B(\bar{u}(s), \bar{u}(s)) + f_0\left(\omega\frac{s}{\varepsilon}\right)\right)\mathrm{d}s. \tag{70}$$

Denote by $\varphi(\tau, x, \omega, \varepsilon)$, $(\bar{\varphi}(\tau, x, \omega, \varepsilon))$ a unique solution of Eq. (69) (respectively (70). According to Theorem 2.8 the cocycle $(\varphi(\cdot,\cdot,\cdot,\varepsilon))$ $(\bar{\varphi}(\cdot,\cdot,\cdot,\varepsilon))$, generated by Eq. (69) (respectively (70)), has an absorbing ball $B[0, R_0]$ $(B[0, \bar{R}_0])$ in E, where $R_0 := \|f\|/\alpha(\bar{R}_0 := \|f_0\|/\alpha)$. This means that for every ball $B[0, R]$ (respectively $B[O, \bar{R}]$) there exists a positive number $L = L(R)$ (respectively $\bar{L} = \bar{L}(\bar{R})$) such that

$$U(t, \omega, \varepsilon)B[0, R] \subseteq B[0, R_0], \tag{71}$$

$$(\bar{U}(t, \omega, \varepsilon)B[0, \bar{R}] \subseteq B[0, \bar{R}_0]) \tag{72}$$

for all $t \geqslant L(t \geqslant \bar{L})$, $\varepsilon \in [0, \varepsilon_0]$ and $\omega \in \Omega$, where $U(t, \omega, \varepsilon) := \varphi(t, \cdot, \omega, \varepsilon)$ $(\bar{U}(t, \omega, \varepsilon) := \bar{\varphi}(t, \cdot, \omega, \varepsilon)$.

According to Theorem 2.8 the cocycle $\varphi(\cdot,\cdot,\cdot,\varepsilon)(\bar{\varphi}(\cdot,\cdot,\cdot,\varepsilon))$ is uniformly bounded for $t \geqslant 0$, that is, for every ball $B[0, R_1](B[0, \bar{R}_1])$ there exists a ball $B[0, R_2](B[0, \bar{R}_2])$ such that

$$U(t, \omega, \varepsilon)B[0, R_1] \subseteq B[0, R_2], \tag{73}$$

$$(\bar{U}(t, \omega, \varepsilon)B[0, \bar{R}_1] \subseteq B[0, \bar{R}_2]) \tag{74}$$

for all $t \geqslant 0, \varepsilon \in [0, \varepsilon_0]$ and $\omega \in \Omega$.

Theorem 5.2. *Let $L > 0$ be arbitrary but fixed. If $\varphi(0, x, \omega, \varepsilon) = \bar{\varphi}(0, x, \omega, \varepsilon) = x \in B[0, \bar{R}_0]$, that is, the initial points coincide and belong to the absorbing ball of Eq. (68) and the condition (39) is fulfilled, then the following relation takes place*

$$\sup\{|\varphi(t, x, \omega, \varepsilon) - \bar{\varphi}(t, x, \omega, \varepsilon)|_E : 0 \leqslant t \leqslant L, |x|_E \leqslant \bar{R}_0, \omega \in \Omega\} \to 0 \tag{75}$$

as $\varepsilon \to 0$.

Proof. The proof below goes along the same lines as the proofs of the corresponding results from [13,17,18]. We set $v(t) := \varphi(t,x,\omega,\varepsilon) - \bar{\varphi}(t,x,\omega,\varepsilon)$. Substracting Eq. (69) from Eq. (70), we obtain

$$v(t) = \int_0^t e^{(t-s)A}(-B(v(s),\varphi(s,x,\omega,\varepsilon)) - B(\bar{\varphi}(s,x,\omega,\varepsilon),v(s)))\mathrm{d}s \\ + \int_0^t e^{(t-s)A} f_1(\omega s)\mathrm{d}s \tag{76}$$

According to Theorem 2.8 $|\varphi(t,x,\omega,\varepsilon)|, |\bar{\varphi}(t,x,\omega,\varepsilon)| \leqslant r_0$ for all $t \geqslant 0$, where $r_0 := \max\left\{\frac{\|f\|}{\alpha}, \frac{\|f_1\|}{\alpha}\right\}$. In view of (76) $v(t)$ satisfies the inequality

$$|v(t)|_E \leqslant \left|\int_0^t e^{(t-s)A}(B(v(s),\varphi(s,x,\omega,\varepsilon)) + B(\bar{\varphi}(s,x,\omega,\varepsilon),v(s)))\mathrm{d}s\right|_E + \\ \left|\int_0^t e^{(t-s)A} f_1\left(\omega\frac{s}{\varepsilon}\right)\mathrm{d}s\right|_E, t \in [0,L].$$

By (27) and (28) we see that the first term on the right-hand side of (77) is less than

$$2r_0 \cdot K \cdot C_B \int_0^t e^{-a(t-s)}(t-s)^{\alpha_1}|v(s)|_E\,\mathrm{d}s. \tag{77}$$

We now show that the second term in (77) tends to 0 as $\varepsilon \to 0$ uniformly w.r.t. $t \in [0,L]$, $|x| \leqslant R_0$ and $\omega \in \Omega$. Integrating by part in s and taking into account the inequalities (11),(12), (16),(17) and (65) we find

$$\left|\int_0^t e^{(t-s)A} f_1\left(\omega\frac{s}{\varepsilon}\right)\mathrm{d}s\right|_E \\ = \left|e^{At}\int_0^t f_1\left(\omega\frac{s}{\varepsilon}\right)\mathrm{d}s + \int_0^t Ae^{(t-s)A}\int_t^s f_1\left(\omega\frac{\tau}{\varepsilon}\right)\mathrm{d}\tau\right|_E \mathrm{d}s \\ \leqslant \left\|e^{A\tau}\right\|_{X\to E}\left|\int_0^t f_1\left(\omega\frac{s}{\varepsilon}\right)\mathrm{d}s\right|_X + \int_0^t \left\|Ae^{A(t-s)}\right\|_{X\to E}\left|\int_t^s f_1\left(\omega\frac{s}{\varepsilon}\right)\mathrm{d}s\right|_X \\ \leqslant Kt^{1-\beta_1}e^{-at}k_1\left(\frac{t}{\varepsilon}\right) + \int_0^t K(t-s)^{1-\beta_2}e^{-a(t-s)}k_1\left(\frac{t-s}{\varepsilon}\right)\mathrm{d}s. \tag{78}$$

Let $\alpha \in [0,1)$, $\nu \in (0,1)$ and $\beta \in [0,2)$. Since

$$t^\alpha k_1\left(\frac{t}{\varepsilon}\right) \leqslant \sup_{0\leqslant t\leqslant \varepsilon^\nu} t^\alpha k_1\left(\frac{t}{\varepsilon}\right) + \sup_{\varepsilon^\nu \leqslant t \leqslant L} t^\alpha k_1(t) \\ \leqslant \varepsilon^{\alpha\nu}k_1(0) + L^\alpha k_1(\varepsilon^{\nu-1}) := c_1(\varepsilon) \to 0 \text{ as } \varepsilon \to 0$$

and

$$\int_0^t s^{1-\beta} k_1\left(\frac{s}{\varepsilon}\right) \mathrm{d}s = \int_0^{\varepsilon^\nu} s^{1-\beta} k_1\left(\frac{s}{\varepsilon}\right) \mathrm{d}s + \int_{\varepsilon^\nu}^t s^{1-\beta} k_1\left(\frac{s}{\varepsilon}\right) \mathrm{d}s$$
$$\leqslant k_1(0)\frac{\varepsilon^{\nu(2-\beta)}}{2-\beta} + k(\varepsilon^{\nu-1})\frac{(t^{2-\beta} - \varepsilon^{(2-\beta)})}{2-\beta}$$
$$\leqslant k_1(0)\frac{\varepsilon^{\nu(2-\beta)}}{2-\beta} + k(\varepsilon^{\nu-1})\frac{(L^{2-\beta} - \varepsilon^{(2-\beta)})}{2-\beta} := c_2(\varepsilon) \quad \text{as } \varepsilon \to 0 \tag{79}$$

uniformly w.r.t. $t \in [0, L]$ and $\omega \in \Omega$, then

$$\sup_{0 \leqslant t \leqslant L} \left| \int_0^t e^{(t-s)A} f_1\left(\omega \frac{s}{\varepsilon}\right) \mathrm{d}s \right|_E \to 0 \quad \text{as } \varepsilon \to 0 \tag{80}$$

Thus,

$$|v(t)|_E \leqslant C(\varepsilon) + D \cdot \int_0^t (t-s)^{-\alpha_1} |v(s)|_E \mathrm{d}s \tag{81}$$

for all $t \in [0, L]$, where $C(\varepsilon) := c_1(\varepsilon) + c_2(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $D := 2R_0 C_B K$.

We now use the known inequality [16, Ch.7]. If

$$u(t) \leqslant a + b\int_0^t (t-s)^{\beta-1} u(s)\mathrm{d}s, \quad 0 < \beta \leqslant 1, \tag{82}$$

then

$$u(t) \leqslant a G_\beta\left([b\Gamma(\beta)]^{1/\beta} t\right), \tag{83}$$

where $G_\alpha(x)$ is a monotone function, while $\Gamma(\beta)$ is a gamma function. In our case we have

$$|v(t)|_E \leqslant C(\varepsilon) G_\beta\left([b\Gamma(\beta)]^{1/\beta} t\right)$$
$$\leqslant C(\varepsilon) G_\beta\left([b\Gamma(\beta)]^{1/\beta} L\right) := d(\varepsilon) \to 0 \quad (\beta := 1 - \alpha_1 \in (0, 1]) \tag{84}$$

as $\varepsilon \to 0$ uniformly w.r.t. $\omega \in \Omega, x \in B[0, R_0]$ and $t \in [0, L]$ for every $L > 0$. The theorem is proved. □

6. THE GLOBAL AVERAGING PRINCIPLE FOR THE NON-AUTONOMOUS NAVIER–STOKES EQUATIONS

Let Ω be a compact metric space, $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω, E be a Banach space and $\langle E, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with fibre E.

A family of nonempty compact sets $\{I_\omega | \omega \in \Omega\}(I_\omega \subset E)$ is called a local compact attractor (local compact forward attractor) if the followings conditions are fulfilled:

$$\text{(i)} \qquad I = \cup\{I_\omega \ : \ \omega \in \Omega\}$$

is compact;

$$\text{(ii)} \qquad \varphi\lambda_0\big(t, I_\omega^{\lambda_0}, \omega\big) = I^{\lambda_0}_{\sigma(t,\omega)}$$

for all $t \in \mathbb{R}_+$ and $w \in \Omega$;

(iii) there exists $R_0 > 0$ such that $I \subset B(0, R_0) := \{x \in E | |x| < R_0\}$ and

$$\lim_{t\to\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, B[0, R_0], \omega), I) = 0$$

$$\text{(respectively } \lim_{t\to\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, B[0, R_0], \omega), I_{\omega t}) = 0)$$

Theorem 6.1. *Let Λ be a compact metric space, E be a Banach space and $\varphi_\lambda (\lambda \in \Lambda)$ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with fibre E. Suppose that the following conditions are fulfilled:*

(i) *the cocycle ϕ_{λ_0} admits a local compact forward attractor,*
(ii) *the following relation takes place*

$$m_L(\lambda) := \sup\{|\varphi_\lambda(t, x, \omega) - \varphi_{\lambda_0}(t, x, \omega)| \ : \ 0 \leqslant t \leqslant L, \omega \in \Omega, |x| \leqslant R_0\} \to 0 \qquad (85)$$

as $\lambda \to \lambda_0$ for every positive number L;

(iii) *every cocycle φ_λ is asymptotically compact.*

Then the next statements are valid:

(a) *there exists a positive number such that for all $\lambda \in B[\lambda_0, \mu] := \{\lambda \in \Lambda : \rho(\lambda, \lambda_0) \leqslant \mu\}$ the cocycle φ_λ admits μ in $B[0, R_0]$ a forward attractor $\{I_\omega : \omega \in \Omega\}$;*

(b) $$\sup_{\omega\in\Omega} \beta\big(I_\omega^\lambda, I_\omega^{\lambda_0}\big) \to 0$$

as $\lambda \to \lambda_0$.

Proof. Let $\rho > 0$ be an arbitrary small number such that$B[I^{\lambda_0}, \rho] \subset B[0, R_0]$. We choose $L = L(\frac{\rho}{3})$ according to the condition

$$\varphi_{\lambda_0}(t, B[0, R_0], \omega) \subset B\left[I^{\lambda_0}_{\omega}, \frac{\rho}{3}\right]$$

for all $\omega \in \Omega$ and $t \geqslant L(\frac{\rho}{3})$. Now we choose $\varepsilon_0 = \varepsilon_0(L)$ so that $m(\lambda) < \rho/3$ for all $\lambda \in B[\lambda_0, \varepsilon_0]$.

Let $t_1 := L$, then we have $\varphi_{\lambda_0}(t_1, x, \omega) \in B\left[I^{\lambda_0}_{\omega t_2}, \frac{\rho}{3}\right]$ and $\varphi_\lambda(t_1, x, \omega) \in B|I^{\lambda_0}_{\omega t_1}$. We take the point $x_1 := \varphi_\lambda(t_1, x, \omega)$ as the initial point and we consider $\varphi_\lambda(t, x_1, \omega t_1)$ on the segment $[0, L]$,

$$\varphi_{\lambda_0}(t, x_1, \omega t_1); \varphi_\lambda(t, x_1, \omega t_1) = \varphi_\lambda(t, \varphi_\lambda(t_1, x, \omega), \omega t_1) = \varphi_\lambda(t + t_1, x, \omega).$$

On this segment $\varphi_\lambda(t, x_1, \omega t_1) \in$ and $\varphi_{\lambda_0}(t, x_1, \omega t_1)$ will diverge by the value less than $\frac{\rho}{3}$. Since $\varphi_{\lambda_0}(t, x_1, \omega t_1) \in B\left[I_{\omega t_1}, \frac{\rho}{3}\right]$, we get $\varphi_\lambda(2t_1, x_1, \omega) \in B\left[I_{\omega 2t_1}, \frac{2\rho}{3}\right]$.

If we take the point $x_2 := \varphi_\lambda(2t_1, x, \omega)$ as the initial one, then we see that the situation is similar to that occurred at the previous step. Repeating this process, we arrive at a conclusion that $\varphi_\lambda(t, x, \omega) \in B\left[I^{\lambda_0}_{\omega t}, \rho\right] \subset B(0, R_0)$ for all $t \geqslant L\left(\frac{\rho}{3}\right)$ and $\omega \in \Omega$. Since the cocycle φ_λ is asymptotical compact then according to Theorem 3.4 and corollary 3.5 it admits a forward attractor $\{I_\lambda : \omega \in \Omega\}$ such that $I^\lambda := \cup\left\{J^\lambda_\omega : \omega \in \Omega\right\} \subseteq B[I^{\lambda_0}, \rho]$ and, consequently, $\beta(I^\lambda, I^{\lambda_0}) \to 0$ as $\lambda \to \lambda_0$.
Below we proved the inclusion $\varphi_\lambda(t, B[0, R_0], \omega) \subseteq B\left[I^{\lambda_0}_{\omega t}, \frac{\rho}{3}\right]$ for all $t \geqslant L$ and $\omega \in \Omega$ and, consequently, we obtain

$$\varphi_\lambda(t, B[0, R_0], \omega_t) \subseteq B\left[I^{\lambda_0}_{\omega}, \frac{\rho}{3}\right] \tag{86}$$

for all $t \geqslant L$ and $\omega \in \Omega$. Taking onto consideration that

$$I^\lambda_\omega = \bigcap_{t \geqslant 0} \overline{\bigcup_{\tau \geqslant t} \varphi_\lambda(\tau, B[0, R_0], \sigma(-\tau, \omega))} \tag{87}$$

from (86) and (87) it follows that $I^\lambda_\omega \subseteq B\left[I^{\lambda_0}_\omega, \rho\right]$ for all $\omega \in \Omega$ and $\lambda \in B[\lambda_0, \varepsilon_0]$ and, consequently, $\sup_{\omega \in \Omega} \beta\left(I^\lambda_\omega, I^{\lambda_0}_\omega\right) \to 0$ as $\lambda \to \lambda_0$. The theorem is proved.

Remark 6.2. 1. The second condition of Theorem 6.1 is fulfilled, for example, if the space E is finite-dimensional and the mapping $\varphi : \mathbb{R}_+ \times E \times \Omega \times \Lambda \to E$, defined by the equality $\varphi(t, x, \omega, \lambda) := \varphi_\lambda(t, x, \omega)$, is continuous.

In fact, if we suppose that it is not true, then there exist $L_0 > 0$, $\lambda_k \to \lambda_0, x_k \in B[0, R_0], t_l \in [0, L_0]$ and $\omega_k \in \Omega$ such that

$$|\varphi_{\lambda_k}(t_k, x_k, \omega_k) - \varphi_{\lambda_0}(t_k, x_k, \omega_k)| \geqslant \varepsilon_0. \tag{88}$$

Since the sets $B[0, R_0], \Omega$ and $[0, L_0]$ are compacts, we can suppose that the sequences $\{x_k\}, \{t_k\}$ and $\{\omega_k\}$ are convergent. Denote by $t_0 := \lim_{k\to\infty} t_k, x_0 := \lim_{k\to\infty} x_k$ and $\lim_{k\to\infty}\omega_k$. Passing to limit in the equality (88) and taking into account the continuity of the mapping φ we obtain $0 \geqslant \varepsilon_0$. The obtained contradiction prove our statement.

2. Under the conditions of Theorem 6.1 if we suppose that the cocyle φ_{λ_0} admits a compact global forward attractor $\{I_\omega^{\lambda_0} : \omega \in \Omega\}$, i.e.,

$$\lim_{t\to\infty} \sup_{\omega\in\Omega} \beta(\varphi_{\lambda_0}(t, B[0, R], \omega), I_{\omega t}) = 0$$

for every $R > 0$, then should be naturally to hope that for the λ sufficiently close to λ_0 the cocycle φ_λ also will admits a compact global forward attractor $\{I_\omega^\lambda : \omega \in \Omega\}$ in the small neighborhood of I^{λ_0}. Unfortunately, generally speaking, this assertion is not true.

In fact, let φ_0 be a cocycle (dynamical system) generated by the equation $x' = -x$ and φ_λ be a cocycle generated by the equation $x' = -x + \lambda x^3 (\lambda > 0)$. It is clear that the cocycle $\varphi_0(\varphi_\lambda)$ admits a compact global attractor $I^0 = \{0\}(I^\lambda = [-\lambda^{-1/2}, \lambda^{-1/2}])$. In the small neighborhood of the attractor $I^0 = \{0\}$ the cocycle φ_φ (for smallλ) admits a local (but not global) attractor $I^\lambda = \{0\}$.

Theorem 6.3. *Let Λ be a compact metric space, $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on the compact metric space Ω, E be a Banach space and $\varphi_\lambda (\lambda \in \Lambda)$ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with fibre E. Suppose that the following conditions are fulfilled:*

(i) *the cocycle φ_{λ_0} admits a compact global forward attractor;*
(ii) *the following relation takes place*

$$m_L(\lambda) := \sup\{|\varphi_\lambda(t, x, \omega)| - \varphi_{\lambda_0}(t, x, \omega)| : 0 \leqslant t \leqslant L, \omega \in \Omega, |x| \leqslant R_0\} \to 0$$

as $\lambda \to \lambda_0$ for every positive number L;

(iii) *every cocycle φ_λ admits a compact global attractor $\{I_\omega^\lambda : \omega \in \Omega\}$;*
(iv) *the set $I := \cup\{I^\lambda : \lambda \in \Lambda\}$ is bounded in E.*

Then the following equality

$$\lim_{\lambda\to\lambda_0} \sup_{\omega\in\Omega} \beta\left(I_\omega^\lambda, I_\omega^{\lambda_0}\right) = 0$$

is fulfilled and, in particularly,

$$\lim_{\lambda\to\lambda_0} \beta(I^\lambda, I^{\lambda 0}) = 0$$

Proof. Suppose that the conditions of the theorem are fulfilled. According to the condition (iv) there exists a positive number R_0 such that $I \subset B(0, R_0)$. Reasoning as in Theorem 6.1 for all $\rho > 0$ we will find a $L = L(\frac{\rho}{3}) > 0$ and $\delta_0 = \delta_0(\rho) > 0$ such that

$$\varphi_\lambda\big(t, I_\omega^\lambda, \omega\big) \subseteq B\big[I_{\omega t}^{\lambda 0}, \rho\big]$$

for all $t \geqslant L$ and $\omega \in \Omega$ and, consequently,

$$I_\omega^\lambda = \varphi\lambda(t, I_\sigma(-t, \omega), \sigma(-t, \omega)) \subseteq B\big[I_{\omega,}^{\lambda_0} \rho\big]$$

for all $\omega \in \Omega$ and $\rho(\lambda, \lambda_0) < \delta_0$. The theorem is proved. □

Lemma 6.4. *[8]. Let Λ be a compact metric space and $\varphi\colon \mathbb{T}_+ \times W \times \Lambda \times \Omega \mapsto W$ verifies the following conditions:*

(i) *φ is continuous;*
(ii) *for every $\lambda \in \Lambda$ the function $\varphi_\lambda = \varphi(\cdot, \cdot, \lambda, \cdot) : \mathbb{T}_+ \times W \times \Omega \mapsto W$ is a continuous cocycle on Ω with the fibre W;*
(iii) *the cocycle $\varphi\lambda$ admits a pullback attractor $\{I_\omega^\lambda | \omega \in \Omega\}$ for every $\lambda \in \Lambda$;*
(iv) *the set $\cup\{I^\lambda | \lambda \in \Lambda\}$ is precompact, where $I^\lambda = \cup\{I_\omega^\lambda | \omega \in \Omega\}$,*

then the following equality

$$\lim_{\lambda \to \lambda_0, \omega \to \omega_0} \beta\left(I_\omega^\lambda, I_{\omega_0}^{\lambda_0}\right) = 0 \tag{89}$$

takes place for every $\lambda_0 \in \Lambda$ and $\omega_0 \in \Omega$ and

$$\lim_{\lambda \to \lambda_0} \beta(I_\lambda I_{\lambda_0}) = 0 \tag{90}$$

for every $\lambda_0 \in \Lambda$.

Lemma 6.5. *[8]. Let the conditions of Lemma 6.4 and additionally the following condition be fulfilled:*

5. for certain $\lambda_0 \in \Lambda$ the application $F\colon \Omega \mapsto C(W)$, defined by equality $F(\omega) = I_\omega^{\lambda 0}$ is continuous, i.e., $\alpha(F(\omega), F(\omega_0)) \to 0 if \omega \to \omega_0$ for every $\omega_0 \in \Omega$, where α is the full metric of Hausdorff, i.e., $\alpha(A, B) =$ $\max\{\beta(A, B), \beta(B, A)\}$.

Then the equality

$$\lim_{\lambda \to \lambda 0} \sup_{\omega \in \Omega} \beta\big(I_\omega^\lambda, I_\omega^{\lambda 0}\big) = 0 \tag{91}$$

takes place.

Theorem 6.6. *[8]. Let W possess the property (S) and let the cocycle φ admit a compact pullback attractor $\{I_\omega|\omega\in\Omega\}$, then:*

(i) *the set I_ω is connected for every $\omega\in\Omega$;*
(ii) *if the space Ω is connected, then the set $I=\cup\{I_\omega|\omega\in\Omega\}$ also is connected.*

Theorem 6.7. *Let $\varepsilon\in(0,\varepsilon_0)$, Ω be compact and connected and $\varphi_\varepsilon(\bar\varphi_\varepsilon)$ be a cocycle generated by the Eq. (60) (respectively, by Eq. (66)).*
Suppose that the following conditions are fulfilled:

(i) $B(\omega):=B_0(\omega)+B_1(\omega)(\omega\in\Omega)$, $B_0, B_1\in C(\Omega, L^2(E,F))$;
(ii) *the bilinear forms B and B_0 satisfy the condition (19);*
(iii) *the average of $B_1(\omega)$ is equal to 0, i.e., $\lim\limits_{t\to\infty}\frac{1}{t}\int_0^t B_1(\omega s)\mathrm{d}s=0$ uniformly w.r.t $\omega\in\Omega$;*
(iv) *the bilinear form B_0 satisfies the condition (39);*
(v) *the cocycles φ_ε and $\bar\varphi_\varepsilon(\varepsilon\in(0,\varepsilon_0]))$ are asymptotically compact.*

Then the following statements are true:

(a) *for every $\varepsilon\in(0,\varepsilon0])$ and $\omega\in\Omega$ the set $I^\varepsilon_\omega:=\{x\in E:$ the solution $\varphi_\varepsilon(t,x,\omega)$ of Eq. (64) is defined and bounded on $\mathbb{R}\}$ (respectively $\bar I^\varepsilon_\omega:=\{x\in E:$ the solution $\bar\varphi_\varepsilon(t,x,\omega)$ of Eq. (66) is defined and bounded on $\mathbb{R}\}$) is nonempty, compact and connected;*
(b) *the cocycle $\varphi_\varepsilon(\bar\varphi_\varepsilon)$ admits a compact global attractor $\{I^\varepsilon_\omega:\omega\in\Omega\}$ $\big($respectively $\{\bar I^\varepsilon_\omega:\omega\in\Omega\}\big)$;*
(c) *the set I^ε (respectively$\bar I^\varepsilon$) is compact and connected;*
(d) *the set $I:=\cup\{I^\varepsilon:\varepsilon\in[0,\varepsilon_0]\}$ $\big($respectively, $\bar I:=\cup\{\bar I^\varepsilon:\varepsilon\in[0,\varepsilon0]\}$, where $\bar I\varepsilon:=\cup\{\bar I^\varepsilon_\omega:\omega\in\Omega\}\big)$ is compact, where $I^\varepsilon:=\cup\{I^\varepsilon_\omega:\omega\in\Omega\}$, $I^0=\bar I^0:=\cup\{\bar I^0_\omega:\omega\in\Omega\}$ and $\{\bar I^0_\omega:\omega\in\Omega\}$ is a compact global attractor of Eq. (66), when $\varepsilon=1$;*
(e) *$\lim\limits_{\varepsilon\to0}\beta(I^\varepsilon,\bar I^0)=0$, where β is a semi-distance of Hausdorff;*
(f) *If a dynamical system $(\Omega,\mathbb{R},\sigma)$ is periodic, i.e., there exists $\omega_0\in\Omega$ such that $\omega_0\tau=\omega_0$ and $\Omega=\{\omega_0 t: t\in[0,\tau)\}$, then*

$$\lim_{\varepsilon\to0}\ \sup\{\beta(I^\varepsilon_\omega, I^0_\omega)\}=0$$

Proof. Let $\varepsilon\in(0,\varepsilon_0)$, Ω be compact and connected and $\varphi_\varepsilon(\bar\varphi\varepsilon)$ be a cocycle generated by the Eq. (60) (respectively, by Eq. (66)), then we have

$$\varphi_\varepsilon(t,x,\omega)=\varphi(\varepsilon t,x,\omega,\varepsilon)(\text{respectively}, \bar\varphi_\varepsilon(t,x,\omega)=\bar\varphi(\varepsilon t,x,\omega,\varepsilon), \tag{92}$$

for all $t\in\mathbb{R}_+, x\in E$ and $\omega\in\Omega$, where $\varphi(\cdot,\cdot,\cdot,\varepsilon)$ (respectively $\bar\varphi(\cdot,\cdot,\cdot,\varepsilon)$ is a cocycle generated by the equation (67) (respectively (68)). From the equality (92) it follows that $\{I^{\varepsilon}_{\omega} : \omega \in \Omega\}$ (respectively $\{\bar{I}^{\varepsilon}_{\omega} : \omega \in \Omega\},$) is a compact global attractor of the equation (67) (respectively (68)). Now to finish the proof of theorem it is sufficient to apply Theorems 5.2, 6.3, 6.6 and Lemmas 6.4, 6.5. The theorem is proved. □

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