EFFECTIVE MACROSCOPIC DYNAMICS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN PERFORATED DOMAINS[*]

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Abstract. An effective macroscopic model for a stochastic microscopic system is derived. The original microscopic system is modeled by a stochastic partial differential equation defined on a domain perforated with small holes or heterogeneities. The homogenized effective model is still a stochastic partial differential equation but defined on a unified domain without holes. The solutions of the microscopic model are shown to converge to those of the effective macroscopic model in probability distribution, as the size of holes diminishes to zero. Moreover, the long time effectivity of the macroscopic system in the sense of *convergence in probability distribution*, and the effectivity of the macroscopic system in the sense of *convergence in energy* are also proved.

Key words. Stochastic PDEs, effective macroscopic model, stochastic homogenization, white noise, probability distribution, perforated domain

AMS subject classifications. 60H15, 86A05, 34D35

1. Introduction.

In recent years there has been explosive growth of activities in multiscale modeling of complex phenomena in many areas, including material science, climate dynamics, chemistry and biology [15, 31]. Stochastic partial differential equations (SPDEs or stochastic PDEs) — evolutionary equations containing noises — arise naturally as mathematical models of multiscale systems under random influences. In fact, the need to include stochastic effects in mathematical modelling of realistic physical phenomena has become widely recognized in, for example, condensed matter physics, climate and geophysical sciences, and materials sciences. But implementing this idea poses some challenges both in theory and in computation [17, 33].

This paper is devoted to the effective macroscopic dynamics of microscopic systems modeled by parabolic SPDEs in perforated media which exhibit small-scale heterogeneities. One example of such microscopic systems of interest is composite materials with microscopic heterogeneities under the impact of external random fluctuations. The heterogeneity scale is taken to be much smaller than the macroscopic scale, which is equivalent, here, to assuming that the heterogeneities are evenly distributed. From a mathematical point of view, one can assume that microscopic heterogeneities (holes) are periodically placed in the media. This periodicity can be represented by a small positive parameter ϵ (i.e., the period). In fact we work on the space-time cylinder $D_\epsilon \times (0, T)$, with $T > 0$, and D_ϵ being the spatial domain obtained by removing a number N_ϵ of holes, of size ϵ, periodically distributed, from a fixed domain D. When taking $\epsilon \to 0$, the holes inside D are smaller and smaller and their numbers goes to ∞. This signifies that the heterogeneities are finer and finer.

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[*] This work was partly supported by the NSF Grants DMS-0209326 & DMS-0542450 and the Outstanding Overseas Chinese Scholars Fund of the Chinese Academy of Sciences.

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There has been a lot of work on the homogenization problem for the deterministic systems defined in such perforated domain or other heterogeneous media, see for example [6, 24, 25, 28, 30] for heat transfer in a composite material, [6, 8, 11] for the wave propagation in a composite material and [21, 23] for the fluid flow in a porous media. For an introduction in homogenization, see [9, 18, 27].

Recently there are also works on homogenization of partial differential equations (PDEs) in the random context; see [19, 22, 26] for PDEs with random coefficients, and [5, 35, 36] for PDEs in randomly perforated domains. And also see a survey book about the homogenization results in a random context [18]. A basic assumption in these texts is the ergodic hypotheses on the coefficients for the passing of the limit of $\epsilon \to 0$. Note that the microscopic models in these works are partial differential equations with random coefficients, so-called random partial differential equations (random PDEs), instead of stochastic PDEs.

In the present paper, the microscopic model is a SPDE defined in a perforated domain. Homogenization techniques are employed to derive an effective, simplified, macroscopic model. Homogenization is a formal mathematic procedure for deriving macroscopic models from microscopic systems. It has been applied to a variety of problems including composite materials modeling, porous media and climate modeling; see [9, 10, 18, 27]. Homogenization provides effective macroscopic behavior of the systems with microscopical heterogeneities for which direct numerical simulations are usually too expensive.

We consider a spatially extended system where stochastic effects are taken into account in the model equation, defined on a deterministic domain but perforated with small scale holes. Specifically, we study a class of stochastic partial differential equations driven by white noise on a perforated domain in the following form

$$du_\epsilon(t) = (\mathcal{A}_\epsilon u_\epsilon + F_\epsilon(x,t))dt + G_\epsilon(x,t)dW(t), \quad 0 < t < T, \epsilon > 0, \tag{1.1}$$

which will be described in detail in the next section. For the general theory of SPDEs we refer to [12]. The goal here is to derive the homogenized equation (effective equation), which is still a stochastic partial differential equation, for (1.1) by the homogenization techniques in the sense of *probability.*

Homogenization theory has been developed for deterministic systems, and compactness discussion for the solutions $\{u_\epsilon\}_\epsilon$ in some function space is a key step in various homogenization approaches [9]. However, due to the appearance of the stochastic term in the above microscopic system considered in this paper, such compactness result does not hold for this stochastic system. Fortunately the compactness in the sense of probability, that is, the tightness of the distributions for $\{u_\epsilon\}$, still holds. So one appropriate way is to homogenize the stochastic system in the sense of probability. The goal in this paper is to derive an effective macroscopic equation for the above microscopic system, by homogenization in the sense of probability. It is shown that the solution u_ϵ of the microscopic or heterogeneous system converges to that of the macroscopic or homogenized system as $\epsilon \downarrow 0$ in probability distribution. This means that the distribution of $\{u_\epsilon\}_\epsilon$ weakly converges, in some appropriate space, to the distribution of a stochastic process which solves the macroscopic effective equation. Moreover, the long time effectivity of the homogenized macroscopic system is demonstrated, that is, the solution $u_\epsilon(t)$ is shown to converge to the stationary solution of the homogenized equation as $t \to \infty$ and $\epsilon \downarrow 0$ in the sense of probability distribution. Furthermore, the effectivity of the macroscopic system in the sense of convergence in energy is also shown.

In our approach, one difficulty is that the spatial domain is changing as $\epsilon \to 0$. To overcome this we use the extension operator introduced in [8] and introduce a new probability space depending on a parameter in which the solution is uniformly bounded. One novelty here is that the original microscopic model is a stochastic PDE, instead of a random PDE as studied by others, e.g., [19, 26, 7].

This paper is organized as follows. The problem formulation is stated in §2. Section 3 is devoted to basic properties of the microscopic system. The effective macroscopic equation is derived in §4. The long time effectivity of the homogenized macroscopic system is considered in §6. Finally, the effectivity of the macroscopic system in the sense of convergence in energy is shown in §5. Moreover, in the Appendix we present the explicit expression of the homogenization matrix.

2. Problem formulation. Let D be an open bounded set in R^n, $n \geq 2$, with smooth boundary ∂D and $\epsilon > 0$ is a small parameter. Let $Y = [0, l_1) \times [0, l_2) \times \cdots \times [0, l_n)$ be a representative (cubic) cell in R^n and S an open subset of Y with smooth boundary ∂S, such that $\overline{S} \subset Y$. Write $l = (l_1, l_2, \cdots, l_n)$. Define $\epsilon S = \{\epsilon y : y \in S\}$. Denote by $S_{\epsilon,k}$ the translated image of ϵS by kl, $k \in Z^n$, $kl = (k_1 l_1, k_2 l_2, \cdots, k_n l_n)$. And let S_ϵ be the set of all the holes contained in D and $D_\epsilon = D \backslash S_\epsilon$. Then D_ϵ is a periodically perforated domain with holes of the same size as period ϵ. We assume that the holes do not intersect with the boundary ∂D, which implies that $\partial D_\epsilon = \partial D \cup \partial S_\epsilon$. See Fig. 1 for the case $n = 2$. This assumption is for avoiding technicalities and the results of our paper will remain valid without this assumption; see [1].

In the sequel we use the notations

$$Y^* = Y \backslash \overline{S}, \quad \vartheta = \frac{|Y^*|}{|Y|}$$

with $|Y|$ and $|Y^*|$ the Lebesgue measure of Y and Y^* respectively. And denote by $\tilde{v}$ the zero extension to the whole D for any function defined on D_ϵ:

$$\tilde{v} = \begin{cases} v & \text{on } D_\epsilon, \\ 0 & \text{on } S_\epsilon. \end{cases}$$

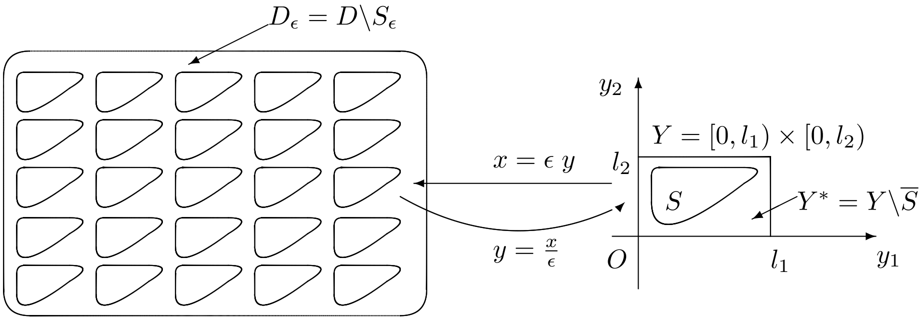

Fig. 1: Geometric setup in R^2

Now for $T > 0$ fixed final time, we consider the following Itô type nonautonomous stochastic partial differential equation defined on the perforated domain D_ϵ in R^n.

$$du_\epsilon(x,t) = \Big(div\big(A_\epsilon(x)\nabla u_\epsilon(x,t)\big) + f_\epsilon(x,t)\Big)dt + g_\epsilon(t)dW(t) \quad in \ \ D_\epsilon \times (0,T), \tag{2.1}$$

$$u_\epsilon = 0 \ \ on \ \ \partial D \times (0,T), \tag{2.2}$$

$$\frac{\partial u_\epsilon}{\partial \nu_{A_\epsilon}} = 0 \ \ on \ \ \partial S_\epsilon \times (0,T), \tag{2.3}$$

$$u_\epsilon(0) = u_\epsilon^0 \ \ in \ \ D_\epsilon, \tag{2.4}$$

where the matrix A_ϵ is

$$A_\epsilon = \Big(a_{ij}\Big(\frac{x}{\epsilon}\Big)\Big)_{ij}$$

and

$$\frac{\partial \cdot}{\partial \nu_{A_\epsilon}} = \sum_{ij} a_{ij}\Big(\frac{x}{\epsilon}\Big)\frac{\partial \cdot}{\partial x_j} n_i$$

with n the exterior unit normal vector on the boundary ∂D_ϵ.

We make the following assumptions on the coefficients:

1. $a_{ij} \in L^\infty(R^n), \ \ i,j = 1, \cdots, n;$
2. $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \alpha \sum_{i=1}^n \xi_i^2$ for $\xi \in R^n$ and α a positive constant;
3. a_{ij} are Y-periodic.

Furthermore we assume that

$$f_\epsilon \in L^2(D_\epsilon \times [0,T]) \tag{2.5}$$

and for $0 \leq t \leq T$, $g_\epsilon(t)$ is a linear operator from ℓ^2 to $L^2(D_\epsilon)$ defined as

$$g_\epsilon(t)k = \sum_{i=1}^\infty g_\epsilon^i(x,t)k_i, \ \ k = (k_1, k_2, \cdots) \in \ell^2$$

where $g_\epsilon^i(x,t) \in L^2(D_\epsilon \times [0,T])$, $i = 1, 2, \cdots$, are measurable functions with

$$\sum_{i=1}^\infty |g_\epsilon^i(x,t)|^2_{L^2(D_\epsilon)} < C_T, \ \ t \in [0,T] \tag{2.6}$$

for some positive constant C_T independent of ϵ. In (2.1), $W(t) = (W_1(t), W_2(t), \cdots)$ is a Wiener process in ℓ^2 with covariance operator $Q = Id_{\ell^2}$ and $\{W_i(t) : i = 1, 2, \cdots\}$ are mutually independent real valued standard Wiener processes on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a canonical filtration $(\mathcal{F}_t)_{t\geq 0}$. Then

$$|g_\epsilon(t)|^2_{\mathcal{L}_2^Q} = \sum_{i=1}^\infty |g_\epsilon^i(x,t)|^2_{L^2(D_\epsilon)} < C_T, \ \ t \in [0,T]. \tag{2.7}$$

Here $\mathcal{L}_2^Q$ is the space of Hilbert-Schmit operators [12, 16]. Denote by $\mathbf{E}$ the expectation operator with respect to $\mathbf{P}$.

The following compactness result [20] will be used in our approach. Let $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$ be three reflective Banach spaces and $\mathcal{X} \subset \mathcal{Y}$ with compact and dense embedding. Define Banach space

$$G = \{v : v \in L^2(0,T;\mathcal{X}), \frac{dv}{dt} \in L^2(0,T;\mathcal{Z})\}$$

with norm

$$|v|_G^2 = \int_0^T |v(s)|_{\mathcal{X}}^2 ds + \int_0^T \left|\frac{dv}{ds}(s)\right|_{\mathcal{Z}}^2 ds, \;\; v \in G.$$

LEMMA 2.1. *If B is bounded in G, then it is precompact in $L^2(0,T;\mathcal{Y})$.*

Let $\mathcal{S}$ be a Banach space and $\mathcal{S}'$ be the strong dual space of $\mathcal{S}$. We recall the definitions and some properties of weak convergence and weak* convergence [34].

DEFINITION 2.2. *A sequence $\{s_n\}$ in $\mathcal{S}$ is said to converge weakly to $s \in \mathcal{S}$ if $\forall s' \in \mathcal{S}'$,*

$$\lim_{n\to\infty} (s', s_n)_{\mathcal{S}',\mathcal{S}} = (s', s)_{\mathcal{S}',\mathcal{S}}$$

which is written as $s_n \rightharpoonup s$ weakly in $\mathcal{S}$. Note that (s', s) denotes the value of the continuous linear functional s' at the point s.

LEMMA 2.3. ***(Eberlein-Shmulyan)*** *Assume that $\mathcal{S}$ is reflexive and let $\{s_n\}$ be a bounded sequence in $\mathcal{S}$. Then there exists a subsequence $\{s_{nk}\}$ and $s \in \mathcal{S}$ such that $s_{nk} \rightharpoonup s$ weakly in $\mathcal{S}$ as $k \to \infty$. If all the weak convergent subsequence of has the same limit s, then the whole sequence $\{s_n\}$ weakly converges to s.*

DEFINITION 2.4. *A sequence $\{s'_n\}$ in $\mathcal{S}'$ is said to converge weakly* to $s' \in \mathcal{S}'$ if $\forall s \in \mathcal{S}$,*

$$\lim_{n\to\infty} (s'_n, s)_{\mathcal{S}',\mathcal{S}} = (s', s)_{\mathcal{S}',\mathcal{S}}$$

which is written as $s'_n \rightharpoonup s'$ weakly* in $\mathcal{S}'$.

LEMMA 2.5. *Assume that the dual space $\mathcal{S}'$ is reflexive and let $\{s'_n\}$ be a bounded sequence in $\mathcal{S}'$. Then there exists a subsequence $\{s'_{nk}\}$ and $s' \in \mathcal{S}'$ such that $s'_{nk} \rightharpoonup s'$ weakly* in $\mathcal{S}'$ as $k \to \infty$. If all the weakly* convergent subsequence of $\{s'_n\}$ has the same limit s', then the whole sequence $\{s'_n\}$ waekly* converges to s'.*

We also use the following definition of the weak convergence of the Borel probability measures on $\mathcal{S}$, for more we refer to [14].

DEFINITION 2.6. *Let $\{\mu_\epsilon\}_\epsilon$ be a family of Borel probability measures on the Banach space $\mathcal{S}$. We say μ_ϵ weakly converges to a Borel measure μ on $\mathcal{S}$ if*

$$\int_{\mathcal{S}} h d\mu_\epsilon \to \int_{\mathcal{S}} h d\mu, \;\; as \;\; \epsilon \downarrow 0,$$

for any $h \in C_b(\mathcal{S})$, the space of bounded continuous functions on $\mathcal{S}$.

In the following, for a fixed $T > 0$, we always denote by C_T a constant independent of ϵ.

3. Basic properties of the microscopic model. In this section we will present some estimates of the solutions of microscopic model (2.1), useful for the tightness result of the distributions of solution processes in some appropriate space.

Let $H = L^2(D)$ and $H_\epsilon = L^2(D_\epsilon)$. Define the following space

$$V_\epsilon = \{u \in H^1(D_\epsilon), u|_{\partial D} = 0\}$$

provided with the norm

$$|v|_{V_\epsilon} = |\nabla_{A_\epsilon} v|_{\oplus_n H_\epsilon} = \Big|\Big(\sum_{j=1}^{n} a_{ij}\Big(\frac{x}{\epsilon}\Big)\frac{\partial v}{\partial x_j}\Big)_{i=1}^{n}\Big|_{\oplus_n H_\epsilon}.$$

This norm is equivalent to the usual $H^1(D_\epsilon)$-norm, with an embedding constant independent of ϵ, due to the assumptions on a_{ij} in the last section. Here $\oplus_n$ denotes the direct sum of the Hilbert spaces with usual direct sum norm. Let

$$\mathcal{D}(\mathcal{A}_\epsilon) = \Big\{v \in V_\epsilon : div(A_\epsilon \nabla v) \in H_\epsilon \ \ and \ \ \frac{\partial v}{\partial \nu_{A_\epsilon}}\Big|_{\partial S_\epsilon} = 0\Big\}$$

and define operator $\mathcal{A}_\epsilon v = div(A_\epsilon \nabla v)$ for $v \in \mathcal{D}(\mathcal{A}_\epsilon)$. Then system (2.1)-(2.4) can be written as the following abstract stochastic evolutionary equation

$$du_\epsilon = (\mathcal{A}_\epsilon u_\epsilon + f_\epsilon)dt + g_\epsilon dW, \ \ u_\epsilon(0) = u_\epsilon^0. \tag{3.1}$$

By the assumptions on a_{ij}, operator $\mathcal{A}_\epsilon$ generates a strongly continuous semigroup $S_\epsilon(t)$ on H_ϵ. Solution of (3.1) can then be written in the mild sense

$$u_\epsilon(t) = S_\epsilon(t)u_\epsilon^0 + \int_0^t S_\epsilon(t-s)f_\epsilon(s)ds + \int_0^t S_\epsilon(t-s)g_\epsilon(s)dW(s) \tag{3.2}$$

And the variational formulation is

$$\begin{aligned}\big(du_\epsilon(t), v\big)_{H_\epsilon^{-1}, V_\epsilon} = \Big(-\int_{D_\epsilon} A_\epsilon(x)\nabla u_\epsilon(x,t)\nabla v(x)dx + \int_{D_\epsilon} f_\epsilon(x,t)v(x)dx\Big)dt + \\ \int_{D_\epsilon} g_\epsilon(x,t)v(x)dW(t), \ \ in \ \ \mathcal{D}'(0,T), \ \ v \in V_\epsilon,\end{aligned} \tag{3.3}$$

with $u_\epsilon(0,x) = u_\epsilon^0(x)$.

For the well-posedness of system (3.1) we have the following result.

Theorem 3.1. *(**Global well-posedness of microscopic model**) Assume that (2.5) and (2.7) hold. Let u_ϵ^0 be a $\big(\mathcal{F}_0, \mathcal{B}(H_\epsilon)\big)$-measurable random variable. Then system (3.1) has a unique mild solution $u \in L^2\big(\Omega, C(0,T;H_\epsilon) \cap L^2(0,T;V_\epsilon)\big)$, which is also a weak solution in the following sense*

$$\begin{aligned}&(u_\epsilon(t), v)_{H_\epsilon} \\ &= (u_\epsilon^0, v)_{H_\epsilon} + \int_0^t (\mathcal{A}_\epsilon u_\epsilon(s), v)_{H_\epsilon} ds + \int_0^t (f_\epsilon, v)_{H_\epsilon} ds + \int_0^t (g_\epsilon dW, v)_{H_\epsilon}\end{aligned} \tag{3.4}$$

for $t \in [0,T)$ and $v \in V_\epsilon$. Moreover if u_ϵ^0 is independent of $W(t)$ with $\mathbf{E}|u_\epsilon^0|^2_{H_\epsilon}, < \infty$, then

$$\mathbf{E}|u_\epsilon(t)|^2_{H_\epsilon} + \mathbf{E}\int_0^t |u_\epsilon(s)|^2_{V_\epsilon} ds \le \mathbf{E}|u_\epsilon^0|^2_{H_\epsilon} + C_T, \ \ for \ \ t \in [0,T], \tag{3.5}$$

and

$$\mathbf{E}\int_0^t |\dot{u}_\epsilon(s)|^2_{H_\epsilon^{-1}} ds \le C_T(\mathbf{E}|u_\epsilon^0|^2_{H_\epsilon}+1), \quad for \ \ t\in[0,T]. \tag{3.6}$$

If further assume that

$$|\nabla_{A_\epsilon} g_\epsilon(t)|^2_{\mathcal{L}_2^Q} = \sum_{i=1}^{\infty} |\nabla_{A_\epsilon} g_\epsilon^i(t)|^2_{\oplus_n H_\epsilon} \le C_T, \quad for\ t\in[0,T] \tag{3.7}$$

and $u_\epsilon^0 \in V_\epsilon$ *with* $\mathbf{E}|u_\epsilon^0|^2_{V_\epsilon} < \infty$*, then*

$$\mathbf{E}|u_\epsilon(t)|^2_{V_\epsilon} + \mathbf{E}\int_0^t |\mathcal{A}_\epsilon u_\epsilon(s)|^2_{H_\epsilon} ds \le \mathbf{E}|u_\epsilon^0|^2_{V_\epsilon} + C_T, \quad for \ \ t\in[0,T]. \tag{3.8}$$

Moreover, system (3.1) is well-posed on $[0,\infty)$ *when*

$$f_\epsilon \in L^2(0,\infty;H_\epsilon), \quad g_\epsilon \in L^2(0,\infty;\mathcal{L}_2^Q). \tag{3.9}$$

Proof. By the assumption (2.7), we have

$$|g_\epsilon(t)|^2_{\mathcal{L}_2^Q} = \sum_{i=1}^{\infty} |g_\epsilon^i(t,x)|^2_{H_\epsilon} < \infty.$$

Then the classical result of [12] yields the local existence of u_ϵ. And applying the stochastic Fubini theorem, it is easy to verify the local mild solution is also a weak solution.

Now we give the following a *priori* estimates which yields the existence of weak solution on $[0,T]$ provide (2.5) and (2.7) hold.

Applying Itô formula to $|u_\epsilon|^2$, we obtain

$$d|u_\epsilon(t)|^2_{H_\epsilon} - 2(\mathcal{A}_\epsilon u_\epsilon, u_\epsilon)_{H_\epsilon} dt = 2(f_\epsilon, u_\epsilon)_{H_\epsilon} dt + 2(g_\epsilon dW, u_\epsilon)_{H_\epsilon} + |g_\epsilon|^2_{\mathcal{L}_2^Q} dt. \tag{3.10}$$

By the assumption on a_{ij}, we see that

$$-(\mathcal{A}_\epsilon u_\epsilon, u_\epsilon)_{H_\epsilon} \ge \lambda |u_\epsilon|^2_{H_\epsilon}$$

for some constant $\lambda > 0$ independent of ϵ. Then integrating (3.10) with respect to t yields

$$\begin{aligned} &|u_\epsilon(t)|^2_{H_\epsilon} + \int_0^t |u_\epsilon|^2_{V_\epsilon} ds \\ &\le |u_\epsilon^0|^2_{H_\epsilon} + \lambda^{-1}|f_\epsilon|^2_{L^2(0,T;H_\epsilon)} + \int_0^t (g_\epsilon dW, u_\epsilon)_{H_\epsilon} ds + \int_0^t |g_\epsilon|^2_{\mathcal{L}_2^Q} ds. \end{aligned}$$

Taking expectation on both sides of the above inequality, we derive (3.5).

In a similar way, application of Itô formula to $|u_\epsilon|^2_{V_\epsilon} = |\nabla_{A_\epsilon} u_\epsilon|^2_{\oplus_n H_\epsilon}$ results in the relation

$$\begin{aligned} &d|u_\epsilon(t)|^2_{V_\epsilon} + 2(\mathcal{A}_\epsilon u_\epsilon, \mathcal{A}_\epsilon u_\epsilon)_{H_\epsilon} dt \\ &= -2(f_\epsilon, \mathcal{A}_\epsilon u_\epsilon)_{H_\epsilon} dt - 2(g_\epsilon dW, \mathcal{A}_\epsilon u_\epsilon)_{H_\epsilon} + |\nabla_{A_\epsilon} g_\epsilon|^2_{\mathcal{L}_2^Q} dt. \end{aligned} \tag{3.11}$$

Integrating both sides of (3.11) and by the Cauchy-Schwarz inequality, it is easily to have

$$\begin{aligned}&|u_\epsilon(t)|^2_{V_\epsilon}+\int_0^t|\mathcal{A}_\epsilon u_\epsilon|^2_{H_\epsilon}ds\\ &\le |u_\epsilon(0)|^2_{V_\epsilon}+|f_\epsilon|^2_{L^2(0,T;H_\epsilon)}-2\int_0^t(g_\epsilon dW,\mathcal{A}_\epsilon u_\epsilon)_{H_\epsilon}ds+\int_0^t|\nabla_{A_\epsilon}g_\epsilon|^2_{\mathcal{L}_2^Q}ds,\end{aligned}$$

Then taking the expectation, we derive (3.8). By (3.3) and the property of the stochastic integral we easily have (3.6).

Thus, by the above estimates, the solution can be extended to $[0,\infty)$ if (3.9) hold. The proof is complete. □

We recall a probability concept. Let z be a random variable taking values in a Banach space $\mathcal{S}$, namely, $z:\ \Omega\to z$. Denote by $\mathcal{L}(z)$ the distribution (or law) of z. In fact, $\mathcal{L}(z)$ is a Borel probability measure on $\mathcal{S}$ defined as [12]

$$\mathcal{L}(z)(A)=\mathbf{P}\{\omega: z(\omega)\in A\},$$

for every event (i.e., a Borel set) A in the Borel $\sigma-$algebra $\mathcal{B}(\mathcal{S})$, which is the smallest $\sigma-$algebra containing all open balls in $\mathcal{S}$.

As stated in §1, for the SPDE (2.1) we aim at deriving an effective equation in the sense of probability. A solution u_ϵ may be regarded as a random variable taking values in $L^2(0,T;H_\epsilon)$. So for a solution u_ϵ of (2.1)-(2.4) defined on $[0,T]$, we focus on the behavior of distribution of u_ϵ in $L^2(0,T;H_\epsilon)$ as $\epsilon\to 0$. For this purpose, the tightness [14] of distributions is needed. Note that the function space changes with ϵ, which is a difficulty for obtaining the tightness of distributions. Thus we will treat $\{\mathcal{L}(u_\epsilon)\}_{\epsilon>0}$ as a family of distributions on $L^2(0,T;H)$ by extending u_ϵ to the whole domain D. Recall that the distribution (or law) of u_ϵ is defined as:

$$\mathcal{L}(u_\epsilon)(A)=\mathbf{P}\{\omega: u_\epsilon(\cdot,\cdot,\omega)\in A\}$$

for Borel set A in $L^2(0,T;H_\epsilon)$. First we define an extension operator P_ϵ in the following lemmas.

In the following we denote by $\mathbf{L}(\mathcal{X},\mathcal{Y})$ the space of bounded linear operator from Banach space $\mathcal{X}$ to Banach space $\mathcal{Y}$.

Lemma 3.2. *There exists a bounded linear operator*

$$\hat{Q}\in\mathbf{L}(H^k(Y^*),H^k(Y)),\quad k=0,1$$

such that

$$|\nabla\hat{Q}v|_{\oplus_n L^2(Y)}\le C|\nabla v|_{\oplus_n L^2(Y^*)},\quad v\in H^1(Y^*)$$

for some constant $C>0$.

For the proof of Lemma 3.2 see [8].

We define an extension operator P_ϵ in terms of the above bounded linear operator $\hat{Q}$ in the following lemma.

Lemma 3.3. *There exists an extension operator*

$$P_\epsilon\in\mathbf{L}\big(L^2(0,T;H^k(D_\epsilon)),L^2(0,T;H^k(D))\big),\quad k=0,1,$$

such that for any $v\in H^k(D_\epsilon)$

1. $P_\epsilon v = v$ *on* $D_\epsilon \times (0,T)$
2. $|P_\epsilon v|_{L^2(0,T;H)} \leq C_T |v|_{L^2(0,T;H_\epsilon)}$
3. $|\nabla_{A_\epsilon}(P_\epsilon v)|_{L^2(0,T;\oplus_n L^2(D))} \leq C_T |\nabla_{A_\epsilon} v|_{L^2(0,T;\oplus_n L^2(D_\epsilon))}$

where C_T *is a constant independent of* ϵ*.*

Proof. For $\varphi \in H^k(D_\epsilon)$, then

$$\varphi_\epsilon(y) = \frac{1}{\epsilon}\varphi(\epsilon\, y)$$

belongs to $H^k(Y_l^*)$ with Y_l^* the translation of Y^* for some $l \in R^n$. Define

$$\hat{Q}_\epsilon \varphi(x) = \epsilon(\hat{Q}\varphi_\epsilon)\big(\frac{x}{\epsilon}\big). \tag{3.12}$$

Now for $\varphi \in L^2(0,T;H^k(D_\epsilon))$,we define

$$(P_\epsilon \varphi)(x,t) = [\hat{Q}_\epsilon \varphi(t,\cdot)]\Big(\frac{x}{\epsilon}\Big) = \epsilon[\hat{Q}\varphi_\epsilon(t,\cdot)]\Big(\frac{x}{\epsilon}\Big).$$

It is known [8] that the operator $P_\epsilon \in \mathbf{L}\big(L^2(0,T;H^k(D_\epsilon)), L^2(0,T;H^k(D))\big)$, $k = 0,1$ and satisfies the conditions (1)-(3) listed in the lemma. This completes the proof. □

REMARK 3.4. *In Lemma 2.1 of [8], the operator* P_ϵ *defined in* $\mathbf{L}\big(L^\infty(0,T;H^k(D_\epsilon)), L^\infty(0,T;H^k(D))\big)$, $k = 0,1,$ *coincides with the operator defined in Lemma 3.3 above.*

REMARK 3.5. *The estimates in Theorem 3.1 for* u_ϵ *also hold for* $P_\epsilon u_\epsilon$*. In fact estimates (3.5) and (3.8) are easily derived due to the property of the operator of* P_ϵ*. Since the operator* P_ϵ *is defined on* $L^2(0,T;H^k(D_\epsilon))$, $k = 0,1$, *we define*

$$P_\epsilon \dot{u}_\epsilon \equiv \mathcal{A}_\epsilon P_\epsilon u_\epsilon + \tilde{f}_\epsilon + \tilde{g}_\epsilon \dot{W}, \ \ on\ D \times (0,T).$$

By the property of P_ϵ *and the estimates of* u_ϵ*, it is easy to see that*

$$P_\epsilon \dot{u}_\epsilon = (P_\epsilon \dot{u_\epsilon}), \ \ in\ D_\epsilon \times (0,T)$$

and

$$\mathbf{E}|P_\epsilon \dot{u}_\epsilon|_{L^2(0,T;H^{-1})} \leq \mathbf{E}|\dot{u}_\epsilon|_{L^2(0,T;H_\epsilon^{-1})}.$$

4. Effective macroscopic model. We now derive the effective macroscopic model for the original model (2.1). Let $u_\epsilon \in L^2(0,T;H_\epsilon)$ be the solution of system (2.1)-(2.4). Then by the estimates in Theorem 3.1, Remark 3.5 and the Chebyshev inequality [12, 14], it is clear that for any $\delta > 0$ there is a bounded set $K_\delta \subset G$ with spaces $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ in Lemma 2.1 (and in the paragraph immediately before it) are replaced by $H_0^1(D)$, H and $H^{-1}(D)$ respectively, such that

$$\mathbf{P}\{P_\epsilon u_\epsilon \in K_\delta\} > 1 - \delta.$$

Thus K_δ is compact in $L^2(0,T;H)$ by Lemma 2.1. Then $\{\mathcal{L}(P_\epsilon u_\epsilon)\}_\epsilon$ is tight in $L^2(0,T;H)$. The Prokhorov Theorem and the Skorohod embedding theorem ([12]) assure that for any sequence $\{\epsilon_j\}$ with $\epsilon_j \to 0$ as $j \to \infty$, there exists a subsequence $\{\epsilon_{j(k)}\}$, random variables $\{\hat{u}_{\epsilon_{j(k)}}\} \subset L^2(0,T;H_{\epsilon_{j(k)}})$ and $u \in L^2(0,T;H)$ defined on a new probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbf{P}})$, such that

$$\mathcal{L}(P_{\epsilon_{j(k)}} \hat{u}_{\epsilon_{j(k)}}) = \mathcal{L}(P_{\epsilon_{j(k)}} u_{\epsilon_{j(k)}})$$

and

$$P_{\epsilon_{j(k)}}\hat{u}_{\epsilon_{j(k)}} \to u \ \ in \ \ L^2(0,T;H) \ \ as \ \ k \to \infty,$$

for almost all $\omega \in \widehat{\Omega}$. Moreover $P_{\epsilon_{j(k)}}\hat{u}_{\epsilon_{j(k)}}$ solves system (2.1)-(2.4) with W replaced by Wiener process $\widehat{W}_k$ defined on probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbf{P}})$ with same distribution as W. The limit u is unique; see [4], p.333. In the following, we will determine the limiting equation (homogenized effective equation) that u satisfies and the limiting equation is independent of ϵ. After this is done we see that $\mathcal{L}(u_\epsilon)$ weakly converges to $\mathcal{L}(u)$ as $\epsilon \downarrow 0$.

We always assume the following conditions

$$\tilde{f}_\epsilon \rightharpoonup f, \ \ weakly \ in \ \ L^2(0,T;H), \text{ as } \epsilon \to 0, \tag{4.1}$$

and

$$\tilde{g}^i_\epsilon \rightharpoonup g^i, \ \ weakly \ in \ \ L^2(0,T;H), \text{ as } \epsilon \to 0. \tag{4.2}$$

Define a new probability space $(\Omega_\delta, \mathcal{F}_\delta, \mathbf{P}_\delta)$ as

$$\Omega_\delta = \{\omega \in \Omega : u_\epsilon(\omega) \in K_\delta\},$$

$$\mathcal{F}_\delta = \{F \cap \Omega_\delta : F \in \mathcal{F}\},$$

and

$$\mathbf{P}_\delta(F) = \frac{\mathbf{P}(F \cap \Omega_\delta)}{\mathbf{P}(\Omega_\delta)}, \ \ for \ \ F \in \mathcal{F}_\delta.$$

Denote by $\mathbf{E}_\delta$ the expectation operator with respect to $\mathbf{P}_\delta$.

Now we restrict the system on the probability space $(\Omega_\delta, \mathcal{F}_\delta, \mathbf{P}_\delta)$. In the following discussion we aim at obtaining $L^2(\Omega_\delta)$ convergence for any $\delta > 0$ which means the convergence in probability [3, 14].

From the estimates (3.5), (3.6), Remark 3.5 and the compact embedding of $G \hookrightarrow L^2(0,T;H)$, there exists a subsequence of u_ϵ in K_δ, still denoted by u_ϵ, such that for a fixed $\omega \in \Omega_\delta$

$$P_\epsilon u_\epsilon \rightharpoonup u \ \ weakly^* \ in \ \ L^\infty(0,T;H) \tag{4.3}$$

$$P_\epsilon u_\epsilon \rightharpoonup u \ \ weakly \ in \ \ L^2(0,T;H^1) \tag{4.4}$$

$$P_\epsilon u_\epsilon \to u \ \ strongly \ in \ \ L^2(0,T;H) \tag{4.5}$$

$$P_\epsilon \dot{u}_\epsilon \rightharpoonup \dot{u} \ \ weakly \ in \ \ L^2(0,T;H^{-1}). \tag{4.6}$$

Define

$$\xi_\epsilon = \Big(\sum_{j=1}^n a_{ij}\Big(\frac{x}{\epsilon}\Big)\frac{\partial u_\epsilon}{\partial x_j}\Big) = A_\epsilon \nabla u_\epsilon$$

which satisfies

$$-div \ \xi_\epsilon = f_\epsilon + g_\epsilon \dot{W} - \dot{u}_\epsilon \ \ in \ \ D_\epsilon \times (0,T), \tag{4.7}$$

$$\xi_\epsilon \cdot n = 0 \ \ on \ \ \partial S_\epsilon \times (0,T). \tag{4.8}$$

By the hypothesis of a_{ij} and the fact that $(\tilde{u}_\epsilon)_\epsilon$ being bounded in $L^2(0,T;H_0^1)$, we have

$$\tilde{\xi}_\epsilon \rightharpoonup \xi \ \ \textit{weakly in} \ \ L^2(0,T;\oplus_n H). \tag{4.9}$$

We make use of Tartar's method of oscillating test functions to determine the limiting equation [9].

Note that

$$\begin{aligned}\int_0^T\int_D \tilde{\xi}_\epsilon \cdot \nabla v\varphi dxdt = \int_0^T\int_D \tilde{f}_\epsilon v\varphi dxdt + \sum_{i=1}^\infty \int_0^T\int_D \tilde{g}_\epsilon^i v dx\varphi dW_i(t) + \\ \int_0^T\int_D P_\epsilon u_\epsilon \chi D_\epsilon \dot{\varphi} v dxdt\end{aligned} \tag{4.10}$$

for all $v \in H_0^1(D)$ and $\varphi \in \mathcal{D}(0,T)$. We pass to the limit in (4.10) as $\epsilon \to 0$. Due to the facts

$$P_\epsilon u_\epsilon \to u \ \text{ strongly in } \ L^2(0,T;H), \tag{4.11}$$

$$\chi_{D_\epsilon} \rightharpoonup \vartheta \ \text{ weakly}^* \ \text{ in } \ L^\infty(D), \tag{4.12}$$

and the estimate

$$\mathbf{E}\Big|\sum_{i=1}^\infty \int_0^T\int_D \tilde{g}_\epsilon^i v dx\varphi dW_i(t)\Big|^2 \le \sum_{i=1}^\infty |\tilde{g}_\epsilon^i|^2_{L^2(0,T;H)}|v\varphi|^2_{L^2(0,T;H)},$$

by the assumption (4.2), we see that

$$\sum_{i=1}^\infty \int_0^T\int_D \tilde{g}_\epsilon^i v dx\varphi dW_i(t) \to \sum_{i=1}^\infty \int_0^T\int_D g^i v dx\varphi dW_i(t), \ \ \textit{in} \ \ L^2(\Omega).$$

Thus letting $\epsilon \to 0$ in (4.10) and since $L^2(\Omega_\delta)$ is a subspace of $L^2(\Omega)$ one finds that in $L^2(\Omega_\delta)$

$$\begin{aligned}\int_0^T\int_D \xi \cdot \nabla v\varphi dxdt = \int_0^T\int_D f v\varphi dxdt + \sum_{i=1}^\infty \int_0^T\int_D g^i v dx\varphi dW_i(t) \\ + \int_0^T\int_D \vartheta u\dot{\varphi} v dxdt.\end{aligned} \tag{4.13}$$

Hence

$$-div\ \xi(x,t) = f(x,t) + g(x,t)\dot{W} - \vartheta\dot{u} \ \ \textit{in} \ \ D\times(0,T). \tag{4.14}$$

In the following we identify the limit ξ. We follow the approach of deterministic case for the elliptic problem with homogeneous Neumann boundary condition [9].

For any $\lambda \in R^n$, let w_λ be the solution of

$$-\sum_{j=1}^n \frac{\partial}{\partial y_j}\Big(\sum_{i=1}^n a_{ij}(y)\frac{\partial w_\lambda}{\partial y_i}\Big) = 0 \ \ \textit{in} \ \ Y^* \tag{4.15}$$

$$w_\lambda - \lambda\cdot y \quad \text{is } Y-\textit{periodic} \tag{4.16}$$

$$\frac{\partial w_\lambda}{\partial \nu_A} = 0 \qquad\qquad \textit{on } \partial S \tag{4.17}$$

and define

$$w_\lambda^\epsilon = \epsilon(\hat{Q}w_\lambda)\Big(\frac{x}{\epsilon}\Big),$$

where $\hat{Q}$ is in Lemma 3.2. Then we have [9],

$$w_\lambda^\epsilon \rightharpoonup \lambda\cdot x \ \ weakly\ \ in\ \ H^1(D), \tag{4.18}$$

$$\nabla w_\lambda^\epsilon \rightharpoonup \lambda \ \ weakly\ \ in\ \ \oplus_n L^2(D). \tag{4.19}$$

Now we define

$$(\eta_j^\lambda(y))_j = \Big(\sum_{i=1}^n a_{ji}(y)\frac{\partial w_\lambda(y)}{\partial y_i}\Big)_j, \ \ y\in Y^*$$

and $(\eta_\epsilon^\lambda)(x) = (\eta_j^\lambda(x/\epsilon))_j = A_\epsilon^t\nabla w_\lambda^\epsilon$. Then

$$-div\ \tilde{\eta}_\epsilon^\lambda = 0 \ \ in\ \ D \tag{4.20}$$

and due to (4.18) and (4.19)

$$\tilde{\eta}_\epsilon^\lambda \rightharpoonup \mathcal{M}_Y(\eta^\lambda) \ \ weakly\ \ in\ \ L^2(D). \tag{4.21}$$

It is easy to see that $\mathcal{M}_Y(\eta^\lambda) = B^t\lambda$ with $B^t = (\beta_{ji})$ a constant matrix which is determined in the appendix.

Using test function $\varphi v w_\lambda^\epsilon$ with $\varphi\in\mathcal{D}(0,T)$, $v\in\mathcal{D}(D)$ in (4.10) and multiplying both sides of (4.20) with $\varphi v P_\epsilon u_\epsilon$, we thus obtain

$$\begin{aligned}
&\int_0^T\int_D \tilde{\xi}_\epsilon\cdot\nabla v\varphi w_\lambda^\epsilon dxdt + \int_0^T\int_{D_\epsilon}\xi_\epsilon\cdot\nabla w_\lambda^\epsilon v\varphi dxdt\\
&\quad -\int_0^T\int_D\tilde{\eta}_\epsilon^\lambda\cdot\nabla v\varphi P_\epsilon u_\epsilon dxdt - \int_0^T\int_D\tilde{\eta}_\epsilon^\lambda\cdot\nabla(P_\epsilon u_\epsilon)v\varphi dxdt\\
&= \int_0^T\int_D\tilde{f}_\epsilon\varphi v w_\lambda^\epsilon dxdt + \sum_{i=1}^\infty\int_0^T\int_D\tilde{g}_\epsilon^i v w_\lambda^\epsilon dx\varphi dW_i(t) + \int_0^T\int_D P_\epsilon u_\epsilon\chi_{D_\epsilon}\dot{\varphi}v w_\lambda^\epsilon dxdt.
\end{aligned}$$

Then by the definition of ξ_ϵ, η_ϵ^λ and the assumptions (4.1), (4.2), using the convergence (4.9), (4.11), (4.12), (4.18), (4.19) and (4.21), we have in $L^2(\Omega_\delta)$

$$\begin{aligned}
&\int_0^T\int_D\xi\cdot\nabla v\varphi\lambda\cdot xdxdt - \int_0^T\int_D B^t\lambda\cdot\nabla v\varphi udxdt\\
&= \int_0^T\int_D f\varphi v\lambda\cdot xdxdt + \sum_{i=1}^\infty\int_0^T\int_D g^i v\lambda\cdot xdx\varphi dW_i(t) + \int_0^T\int_D\vartheta uv\dot{\varphi}\lambda\cdot xdxdt.
\end{aligned}$$

That is

$$\begin{aligned}
&\int_0^T\int_D\xi\cdot\nabla(v\lambda\cdot x)\varphi dxdt - \int_0^T\int_D\xi\cdot\lambda v\varphi dxdt - \int_0^T\int_D B^t\lambda\cdot\nabla v\varphi udxdt\\
&= \int_0^T\int_D f\varphi v\lambda\cdot xdxdt + \sum_{i=1}^\infty\int_0^T\int_D g^i v\lambda\cdot xdx\varphi dW_i(t) + \int_0^T\int_D\vartheta uv\dot{\varphi}\lambda\cdot xdxdt.
\end{aligned}$$

Then by using (4.13) with the test function replaced by $v\lambda \cdot x\varphi$ one has

$$\int_0^T \int_D \xi \cdot \lambda v\varphi dxdt = \int_0^T \int_D B^t\lambda \cdot \nabla u\varphi v dxdt$$

which yields

$$\xi \cdot \lambda = B^t\lambda \cdot \nabla u = B\nabla u \cdot \lambda.$$

Then

$$\xi = B\nabla u$$

since λ is arbitrary. Then u satisfies the following equation

$$\vartheta du = \big(div(B\nabla u) + f\big)dt + gdW(t). \tag{4.22}$$

Assume that

$$\tilde{u}_\epsilon^0 \rightharpoonup u^0, \ \ \textit{weakly in } H, \text{ as } \epsilon \to 0. \tag{4.23}$$

We now determine the initial value by suitable test-functions. In fact, taking $v \in \mathcal{D}(D)$ and $\varphi \in \mathcal{D}([0,T])$ with $\varphi(T) = 0$, we have

$$\begin{aligned}\int_0^T \int_D \tilde{\xi}_\epsilon \cdot \nabla v\varphi dxdt = &\int_0^T \int_D \tilde{f}_\epsilon v\varphi dxdt + \sum_{i=0}^{\infty} \int_0^T \int_D \tilde{g}\epsilon^i v dx \varphi dW_i(t) \\ &- \int_0^T \int_D \tilde{u}_\epsilon v\dot{\varphi} dxdt + \int_D \tilde{u}_\epsilon^0 \varphi(0) v dx.\end{aligned}$$

Now let $\epsilon \to 0$, noticing that

$$\begin{aligned}\int_0^T \int_D \tilde{u}_\epsilon v\dot{\varphi} dxdt = \int_0^T \int_D \chi_{D\epsilon} P_\epsilon \tilde{u}_\epsilon v\dot{\varphi} dxdt \to &\int_0^T \int_D \vartheta u v\dot{\varphi} dxdt = \\ - \int_0^T \int_D \vartheta \dot{u} v\varphi dxdt + &\int_D \vartheta u(0)\varphi(0) v dx\end{aligned}$$

by (4.14), we have

$$u(0) = \frac{u^0}{\vartheta}.$$

Here one should notice that the above result is in the sense of $L^2(\Omega_\delta)$. Then the above analysis yields the following results

$$\lim_{\epsilon \to 0} \mathbf{E}_\delta |P_\epsilon u_\epsilon - u|^2_{L^2(0,T;H)} = 0, \tag{4.24}$$

and

$$\lim_{\epsilon \to 0} \mathbf{E}_\delta \int_0^T \int_D (\mathcal{A}_\epsilon P_\epsilon u_\epsilon - B\nabla u) v\varphi dxdt = 0, \tag{4.25}$$

for any $v \in \mathcal{D}(D)$ and $\varphi \in \mathcal{D}([0,T])$.

Now we are in the position to present the homogenized effective equation in the following theorem.

THEOREM 4.1. *(**Effective macroscopic model**) For any $T > 0$, assume that (4.1), (4.2) and (4.23) hold. Let u_ϵ be the solution of (2.1)-(2.4). Then the distribution $\mathcal{L}(P_\epsilon u_\epsilon)$ converges weakly to μ in the space of probability measures on $L^2(0,T;H)$ as $\epsilon \downarrow 0$, with μ being the distribution of u, which is the solution of the following homogenized effective stochastic partial differential equation*

$$\vartheta du = \big(div(B\nabla u) + f\big)dt + gdW(t) \ \ in\ D \times (0,T), \tag{4.26}$$

$$u = 0 \ \ on\ \partial D \times (0,T), \tag{4.27}$$

$$u(x,0) = \frac{u^0}{\vartheta} \ \ in\ D, \tag{4.28}$$

where the constant coefficient $\vartheta = \frac{|Y^*|}{|Y|}$ is defined in the beginning of §2, and the effective matrix $B = (\beta_{ij})$ is determined by (7.4) in Appendix at the end of this paper. Moreover, the coefficients f, g and initial datum u^0 are defined in (4.1), (4.2) and (4.23), respectively.

REMARK 4.2. *This theorem implies that the macroscopic model (4.26) is an effective approximation for the microscopic model (2.1), on any finite time interval $0 < t < T$, in the sense of probability distribution. In other words, if we intend to numerically simulate the microscopic model up to finite time, we could use the macroscopic model as an approximation when ϵ is sufficiently small.*

REMARK 4.3. *Due to the appearance of the stochastic integral term (see (4.10)), this theorem on weak convergence of probability measures does not follow from the deterministic homogenization results and the mild formulation (3.2).*

REMARK 4.4. *The stochastic PDE (4.26) is defined on the homogenized domain D. By the analysis in [12], for any fixed $T > 0$, the macroscopic system (4.26)-(4.28) is well-posed, as long as $f \in L^2(0,T;H)$ and $g \in L^2(0,T;\mathcal{L}_2^Q)$.*

Proof. Noticing the arbitrariness of δ, this is a direct result of the analysis of the first part in this section by the Skorohod theorem and the $L^2(\Omega_\delta)$ convergence of $P_\epsilon u_\epsilon$ on $(\Omega_\delta, \mathcal{F}_\delta, \mathbf{P}_\delta)$. □

We finish this section by the following remark.

REMARK 4.5. *Note that there are several papers on effective dynamics for partial differential equations with random coefficients (so called random PDEs; not stochastic PDEs); see [19, 26, 32] and reference therein. In [19, 26], a random partial differential equation is obtained as the homogenized effective equation for a random system with fast or small scales on time or spatial variable. And the distribution of solution of heterogeneous system converges weakly to that of homogenized equation. However in [32], the effective equation is obtained as an averaged deterministic equation for a random system with fast scales in time. And the fluctuation of the solution of the random equation around the solution of the averaged equation converges to a generalized Ornstein-Uhlenbeck process in distribution. In the present paper, the original microscopic model is a stochastic PDE (i.e., PDE with white noise) and the effective macroscopic equation is still a stochastic partial differential equation.*

5. Long time effectivity of the macroscopic model. In this section we consider the long time effectivity of the homogenized system (4.26) in the autonomous case, i.e., when f_ϵ and g_ϵ (and thus f and g) are independent of time t. It is proved in section 4 that for fixed $T > 0$ the macroscopic behavior of the microscopic system (2.1)-(2.4) can be approximated by the macroscopic model (4.26) in the sense of probability distribution. In fact we can show the long time approximation. More specifically, we now prove that in the sense of distribution, all solutions of (2.1)-(2.4) converge to the unique stationary solution of (4.26) as $T \to \infty$ and $\epsilon \to 0$, under the assumption that $f_\epsilon \in H_\epsilon$ and $g_\epsilon^i \in V_\epsilon$ are *independent* of time t and

$$\sum_{i=1}^{\infty} |\nabla_{A_\epsilon} g_\epsilon^i(x)|^2_{\oplus_n H_\epsilon} < C^*. \tag{5.1}$$

Here C^* is a positive constant independent of ϵ.

By the above assumptions, as well as the properties of a_{ij} and β_{ij}, a standard argument (see [13], Section 6) yields that the system (3.1) and (4.26) have unique stationary solutions $u_\epsilon^*(x,t)$ and $u^*(x,t)$, defined for $t > 0$. We denote by μ_ϵ^* and μ^* the distributions of $P_\epsilon u_\epsilon^*$ and u^* in the space H, respectively. Then if $\mathbf{E}|u_\epsilon^0|^2 < \infty$ and $\mathbf{E}|u^0|^2 < \infty$,

$$\Big| \int_H h d\mu_\epsilon(t) - \int_H h d\mu_\epsilon^* \Big| \le C(u_\epsilon^0) e^{-\gamma t}, \ \ t > 0, \tag{5.2}$$

$$\Big| \int_H h d\mu(t) - \int_H h d\mu^* \Big| \le C(u^0) e^{-\gamma t}, \ \ t > 0, \tag{5.3}$$

for some constant $\gamma > 0$ and any $h : H \to R^1$ with $\sup |h| \le 1$ and $\mathrm{Lip}(h) \le 1$. Here $\mu_\epsilon(t) = \mathcal{L}(P_\epsilon u_\epsilon(t, u_\epsilon^0))$, $\mu(t) = \mathcal{L}(u(t, \frac{u^0}{\vartheta}))$, and $C(u_\epsilon^0)$ and $C(u^0)$ are positive constants depending only on the initial value u_ϵ^0 and u^0 respectively. The above convergence also yields that $\mu_\epsilon(t)$ and $\mu(t)$ weakly converges to μ_ϵ^* and μ^* respectively, as $t \to \infty$.

We will give some additional a *priori* estimates which is uniform with respect to ϵ to ensure the tightness of the stationary distributions. For Banach space U and $p > 1$, we define $W^{1,p}(0,T;U)$ as the space of functions $h \in L^p(0,T;U)$ such that

$$|h|^p_{W^{1,p}(0,T;U)} = |h|^p_{L^p(0,T;U)} + \Big| \frac{dh}{dt} \Big|^p_{L^p(0,T;U)} < \infty.$$

And for any $\alpha \in (0,1)$, define $W^{\alpha,p}(0,T;U)$ as the space of function $h \in L^p(0,T;U)$ such that

$$|h|^p_{W^{\alpha,p}(0,T;U)} = |h|^p_{L^p(0,T;U)} + \int_0^T \int_0^T \frac{|h(t) - h(s)|^p_U}{|t-s|^{1+\alpha p}} ds dt < \infty.$$

For $\rho \in (0,1)$, we denote by $C^\rho(0,T;U)$ the space of functions $h : [0,T] \to \mathcal{X}$ that are Hölder continuous with exponent ρ.

In the remaining part of this section, we always assume that f_ϵ and g_ϵ^i are independent of time t with (5.1) hold. And for $T > 0$ denote by $\mathbf{u}^*_{\epsilon,T}$ (respectively, $\mathbf{u}^*_T$) the distribution of stationary process $P_\epsilon u_\epsilon^*(\cdot)$ (respectively, $u^*(\cdot)$) in the space $L^2(0,T;H^1)$. Then we have the following result.

Lemma 5.1. *For any $T > 0$ the family $\mathbf{u}^*_{\epsilon,T}$ is tight in the space $L^2(0,T;H^{2-\iota})$ with $\iota > 0$.*

Proof. Since u^*_ϵ is stationary, by (3.8), we see that

$$\mathbf{E}|u^*_\epsilon|^2_{L^2(0,T;H^2_\epsilon)} < C_T. \tag{5.4}$$

Now represent u^*_ϵ in the form

$$u^*_\epsilon(t) = u^*_\epsilon(0) + \int_0^t \mathcal{A}_\epsilon u^*_\epsilon(s)ds + \int_0^t f_\epsilon(x)ds + \int_0^t g_\epsilon(x)dW(s).$$

Also by the stationarity of u^*_ϵ and (3.8) we obtain

$$\mathbf{E}\Big|\int_0^t \mathcal{A}_\epsilon P_\epsilon u^*_\epsilon(s)ds + \int_0^t \tilde{f}_\epsilon(x)ds\Big|^2_{W^{1,2}(0,T;H)} \le C_T. \tag{5.5}$$

Let $M_\epsilon(s,t) = \int_s^t \tilde{g}_\epsilon(x)dW(s)$. By Lemma 7.2 of [12] and Hölder inequality, we derive that

$$\begin{aligned}\mathbf{E}|M_\epsilon(s,t)|^4_{V_\epsilon} \le c\Big(\int_s^t |\nabla_{A_\epsilon}\tilde{g}_\epsilon(x)|^2_{\mathcal{L}_2^Q}d\tau\Big)^2 &\le K(t-s)\int_s^t |\nabla_{A_\epsilon}\tilde{g}_\epsilon(x)|^4_{\mathcal{L}_2^Q}d\tau \\ &\le KC^{*2}|t-s|^2\end{aligned}$$

for $t \in [s,T]$, where K is a positive constant independent of ϵ, s and t. Then

$$\mathbf{E}\int_0^T |M_\epsilon(0,t)|^4_{V_\epsilon}dt \le C_T \tag{5.6}$$

and

$$\mathbf{E}\int_0^T\int_0^T \frac{|M_\epsilon(0,t) - M_\epsilon(0,s)|^4_{V_\epsilon}}{|t-s|^{1+4\alpha}}dsdt \le C_T. \tag{5.7}$$

Combining (5.4)-(5.7), and the compact embedding of

$$L^2(0,T;H^2) \cap W^{1,2}(0,T;H) \subset L^2(0,T;H^{2-\iota})$$

and

$$L^2(0,T;H^2) \cap W^{\alpha,4}(0,T;H^1) \subset L^2(0,T;H^{2-\iota})$$

we obtain the tightness of $\mathbf{u}^*_{\epsilon,T}$. This completes the proof. □

The above lemma directly yields the following result

Corollary 5.2. *The family $\{\mu^*_\epsilon\}$ is tight in the space H^1.*

By Lemma 5.1, for any fixed $T > 0$, the Skorohod embedding theorem asserts that for any sequence $\{\epsilon_n\}_n$ with $\epsilon_n \to 0$ as $n \to \infty$, there is subsequence $\{\epsilon_{n(k)}\}_k$, a new probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbf{P}})$ and random variables $\overline{u}^*_{\epsilon_{n(k)}} \in L^2(0,T;V_\epsilon)$, $\overline{u}^* \in L^2(0,T;H^1)$ such that

$$\mathcal{L}(P_\epsilon \overline{u}^*_{\epsilon_{n(k)}}) = \mathbf{u}^*_{\epsilon_{n(k)},T}, \ \ \mathcal{L}(\overline{u}^*) = \mathbf{u}^*_T$$

and

$$\overline{u}^*_{\epsilon_{n(k)}} \to \overline{u}^*, \ \ in \ \ L^2(0,T;H^1) \ \ as \ \ k \to \infty.$$

Moreover $\overline{u}^*_{\epsilon_{n(k)}}$ (respectively, $\overline{u}^*$) is the unique stationary solution of equation (3.1) (respectively, (4.26)) with W replaced by $\overline{W}_k$ (respectively, $\overline{W}$). $\overline{W}_k$ and $\overline{W}$ are some Wiener processes defined on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbf{P}})$ with same distribution as W. Then by the analysis of section 4 and the uniqueness of the invariant measure

$$\mathbf{u}^*_{\epsilon,T} \rightharpoonup \mathbf{u}^*_T, \ \ as \ \ \epsilon \to 0$$

for any $T > 0$.

To show the long time effectivity, let $u_\epsilon(t)$, $t \geq 0$, be a weak solution of system (2.1)-(2.4) and define $u^t_\epsilon(\cdot) = u_\epsilon(t + \cdot)$ which is in the space $L^2_{loc}(R_+; V_\epsilon)$ by Theorem 3.1. Then by (5.2)

$$\mathcal{L}(P_\epsilon u^t_\epsilon(\cdot)) \rightharpoonup \mathcal{L}(P_\epsilon u^*_\epsilon(\cdot)), \ \ t \to \infty$$

in the space of probability measures on $L^2_{loc}(R_+; H^1)$. Having the above analysis we draw the following result which implies the long time effectivity of the homogenized effective equation (4.26).

Theorem 5.3. ***(Long time effectivity of macroscopic model)*** *Assume that $f_\epsilon \in H_\epsilon$ and $g^i_\epsilon \in V_\epsilon$ are independent of time t with (5.1) being satisfied, and further assume that (4.1) and (4.2) hold in H. Denote by $u_\epsilon(t)$, $t \geq 0$, the solution of (2.1)-(2.4) and u^* the unique stationary solution of (4.26). Then*

$$\lim_{\epsilon \downarrow 0} \lim_{t \to \infty} \mathcal{L}(P_\epsilon u^t_\epsilon(\cdot)) = \mathcal{L}(u^*(\cdot)), \tag{5.8}$$

where the limits are understood in the sense of weak convergence of Borel probability measures in the space $L^2_{loc}(R_+; H^1)$. That is, the solution of (2.1)-(2.4) converges to the stationary solution of (4.26) in probability distribution as $t \to \infty$ and $\epsilon \to 0$.

Remark 5.4. *This theorem implies that the macroscopic model (4.26) is an effective approximation for the microscopic model (2.1), on very long time scale. In other words, if we intend to numerically simulate the long time behavior of the microscopic model, we could just simulate the macroscopic model as an approximation when ϵ is sufficiently small.*

6. Effectivity in energy convergence . In the last two sections, we have considered finite time and long time effectivity of the macroscopic model (4.26), in the sense of convergence in probability distribution. In this section we focus on the finite time effectivity of the macroscopic model (4.26), but in the sense of convergence in energy. Namely, we show that the solution of the microscopic model (2.1) or (3.1), converges to the solution of the macroscopic model (4.26), in an energy norm.

Let u_ϵ be a weak solution of (3.1) and u be a weak solution of (4.26). We introduce the following energy functionals:

$$\mathcal{E}^\epsilon(u_\epsilon)(t) = \frac{1}{2}\mathbf{E}|\tilde{u}_\epsilon|^2_H + \mathbf{E}\int_0^t \int_D \chi_{D_\epsilon} A_\epsilon \nabla\big(P_\epsilon u_\epsilon(x,\tau)\big)\nabla\big(P_\epsilon u_\epsilon(x,\tau)\big) dx d\tau \tag{6.1}$$

and

$$\mathcal{E}^0(u)(t)=\frac{1}{2}\mathbf{E}|u|_H^2+\mathbf{E}\int_0^t\int_D B\nabla u(x,\tau)\nabla u(x,\tau)dxd\tau. \tag{6.2}$$

By the Itó formula, it is clear that

$$\mathcal{E}^\epsilon(u_\epsilon)(t)=\frac{1}{2}\mathbf{E}|\tilde{u}_\epsilon^0|_H^2+\mathbf{E}\int_0^t\int_D\tilde{f}_\epsilon(x,\tau)\tilde{u}_\epsilon(x,\tau)dxd\tau+\frac{1}{2}\mathbf{E}\int_0^t|\tilde{g}_\epsilon(x,\tau)|^2_{\mathcal{L}_2^Q}d\tau$$

and

$$\mathcal{E}^0(u)(t)=\frac{1}{2}\mathbf{E}|u^0|_H^2+\mathbf{E}\int_0^t\int_D f(x,\tau)u(x,\tau)dxd\tau+\frac{1}{2}\mathbf{E}\int_0^t|g(x,\tau)|^2_{\mathcal{L}_2^Q}d\tau.$$

Then we have the following result on effectivity of the macroscopic model in the sense of convergence in energy.

THEOREM 6.1. ***(Effectivity in energy convergence)*** *Assume that (4.1) and (4.2) hold. If*

$$\tilde{u}_\epsilon^0\to u^0,\ \ \textit{strongly in } H, \textit{as } \epsilon\to 0,$$

then

$$\mathcal{E}^\epsilon(u_\epsilon)\to\mathcal{E}^0(u)\quad in\ \ C([0,T]), \textit{as } \epsilon\to 0.$$

Proof. By the analysis of §4, for any $\delta>0$, $u_\epsilon\to u$ strongly in $L^2(0,T;H)$ on Ω_δ, then by the arbitrariness of δ, it is easy to see that

$$\mathbf{E}\int_0^t\int_D\tilde{f}_\epsilon(x,\tau)\tilde{u}_\epsilon(x,\tau)dxd\tau\to\mathbf{E}\int_0^t\int_D f(x,\tau)u(x,\tau)dxd\tau,\ \ for\ t\in[0,T].$$

Then by $\tilde{g}_\epsilon\rightharpoonup g$ weakly in $L^2(0,t;\mathcal{L}_2^Q)$, we have

$$\mathcal{E}^\epsilon(u_\epsilon)(t)\to\mathcal{E}^0(u)(t)\ \ for\ \ any\ \ t\in[0,T]. \tag{6.3}$$

We now only need to show that $\{\mathcal{E}^\epsilon(u_\epsilon)(t)\}_\epsilon$ is equicontinuous, as then the Ascoli-Arzela's theorem [14] will imply the result in the theorem.

In fact, given any $t\in[0,T]$, and $h>0$ small enough, we have

$$\begin{aligned}&|\mathcal{E}^\epsilon(u_\epsilon)(t+h)-\mathcal{E}^\epsilon(u_\epsilon)(t)|\\ \le&\left|\mathbf{E}\int_t^{t+h}\int_D\tilde{f}_\epsilon(x,\tau)\tilde{u}_\epsilon(x,\tau)dxd\tau\right|+\mathbf{E}\int_t^{t+h}|\tilde{g}_\epsilon(x,\tau)|^2_{\mathcal{L}_2^Q}d\tau\\ \le&\mathbf{E}\Big\{|\tilde{f}_\epsilon|_{L^2(0,T;H)}\int_t^{t+h}|\tilde{u}_\epsilon(x,\tau)|_H^2dxd\tau\Big\}+\mathbf{E}\int_t^{t+h}|\tilde{g}_\epsilon(x,\tau)|^2_{\mathcal{L}_2^Q}d\tau.\end{aligned}$$

Noting that $\tilde{u}_\epsilon\in L^2(0,T;H)$ a.s. and (2.7), we have

$$|\mathcal{E}^\epsilon(u_\epsilon)(t+h)-\mathcal{E}^\epsilon(u_\epsilon)(t)|\to 0,\ \ as\ h\to 0,$$

uniformly on ϵ, which means the equi-continuity of the family $\{\mathcal{E}^\epsilon(u_\epsilon)\}_\epsilon$. This completes the proof. □

7. Appendix: The homogenized matrix. In this Appendix, we give the explicit expression of the homogenized matrix B; for more details see [9]. Let χ^i, $i = 1, \cdots, n$ be the solutions of

$$-\sum_{l,k=1}^{n} \frac{\partial}{\partial y_l}\Big(a_{kl}\frac{\partial(\chi^i - y_i)}{\partial y_k}\Big) = 0 \ \ in \ Y^* \tag{7.1}$$

$$\sum_{l,k=1}^{n} a_{kl}\frac{\partial(\chi^i - y_i)}{\partial y_k} n_l = 0 \ \ on \ \partial S \tag{7.2}$$

$$\chi^i \ \text{is} \ Y - periodic. \tag{7.3}$$

It is easy to calculate that $\chi^i = -w_{e_i} + e_i$ with $\{e_i\}_{i=1}^n$ the canonical basis of R^n. Then

$$\beta_{ij} = \frac{1}{|Y|}\int_Y \sum_{k=1}^{n} a_{kj}\frac{\partial w_{e_i}}{\partial y_k}dy = \frac{1}{|Y|}\int_Y a_{ij}dy - \frac{1}{|Y|}\int_Y \sum_{k=1}^{n} a_{kj}\frac{\partial \chi_i}{\partial y_k}dy. \tag{7.4}$$

Moreover the operator $B = (\beta_{ij})$ satisfies the uniform ellipticity condition: there is a constant $b > 0$ such that

$$\sum_{i,j=1}^{n} \beta_{ij}\xi_i\xi_j \geq b\sum_{i=1}^{n}\xi_i^2, \ \ for \ \xi = (\xi_1, \cdots, \xi_n) \in R^n.$$

Acknowledgments. The authors thank the referees for very helpful suggestions and comments.

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