

STOCHASTIC DYNAMICS OF A COUPLED ATMOSPHERE–OCEAN MODEL

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The investigation of the coupled atmosphere-ocean system is not only scientifically challenging but also practically important.

We consider a coupled atmosphere-ocean model, which involves hydrodynamics, thermodynamics, and random atmospheric dynamics due to short time influences at the air–sea interface. We reformulate this model as a random dynamical system. First, we have shown that the asymptotic dynamics of the coupled atmosphere-ocean model is described by a random climatic attractor. Second, we have estimated the atmospheric temperature evolution under oceanic feedback, in terms of the freshwater flux, heat flux and the external fluctuation at the air–sea interface, as well as the earth’s longwave radiation coefficient and the shortwave solar radiation profile. Third, we have demonstrated that this system has finite degree of freedom by presenting a finite set of determining functionals in probability. Finally, we have proved that the coupled atmosphere-ocean model is ergodic under suitable conditions for physical parameters and randomness, and thus for any observable of the coupled atmosphere-ocean flows, its time average approximates the statistical ensemble average, as long as the time interval is sufficiently long.

Keywords: Stochastic geophysical flow models; random attractor; climate dynamics; finite degrees of freedom; ergodicity.

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1. Geophysical Background

The coupled atmosphere-ocean system defines the environment we live. Randomness or uncertainty is ubiquitous in this coupled, complex, multiscale system: For example, stochastic forcing (wind stress, heat flux and freshwater flux), uncertain parameters, random sources or inputs, and random boundary conditions.

Mathematical models are a key component of our understanding of climate and geophysical systems. It is our belief that the fidelity of these models to nature can greatly benefit through the inclusion of stochastic effects. Taking stochastic effects into account is of central importance for the development of mathematical models of many phenomena in geophysical and climate flows.

We consider a two-dimensional coupled atmosphere-ocean model in the latitude-depth plane, with atmospheric dynamics highly simplified, i.e. the atmospheric dynamics is described by an energy balance model. The oceanic dynamics is described by the Navier–Stokes equation in vorticity form and the transport equations for heat and salinity. The energy balance model is under random impact due to, for example, eddy transport fluctuation, stormy bursts of latent heat, and flickering cloudiness variables. So this coupled atmosphere-ocean model consists of stochastic and deterministic partial differential equations, together with air–sea flux or Neumann boundary conditions. We will reformulate this model as a random dynamical system.

The ocean and the atmosphere are constantly interacting through the air–sea exchange process. The ocean moves much slower than the atmosphere does. It is generally believed that the ocean plays an important role in the global climate dynamics in relatively long time scales, due to ocean’s large capacity of holding and transporting huge amount of heat or cold around the globe [26]. However, a complete quantitative understanding or estimate for ocean’s impact on climate is lacking. A particular issue is: How does the ocean affect or provide feedback to the air temperature, which is the most important climate quantity we are usually concerned about? This is a direct impact of the ocean on the climate. It is desirable to predict or estimate this feedback in the context of our simple coupled atmosphere-ocean model.

The existence and interpretation of climatic attractors have been controversial and have caused a lot of debate [18]. A low-dimensional climatic attractor was regarded as an indication that the main feature of long-time climatic evolution may be viewed as the manifestation of a *deterministic* dynamics. We will consider *random* climate attractors, and the long time regimes that such attractors still carry the stochastic information of the geophysical flow system. We will also investigate the finite dimensionality of the asymptotic dynamics by checking the *determining functionals in probability*.

In a special case of physical parameters and random noise, we obtain a random attractor which is defined by a single random variable. This random variable attracts all other motions exponentially fast. This random variable corresponds to

a unique invariant measure, which is the expectation of the Dirac measure with the random variable as the random mass point; see [2]. In this case, the coupled atmosphere-ocean model is *ergodic*, and thus for any observable of the coupled atmosphere-ocean flows, its time average approximates the statistical ensemble average, as long as the time interval is sufficiently long.

In the next section, we present the coupled atmosphere-ocean model, and discuss the well-posedness of this coupled model in Sec. 3. Then we investigate the dissipativity property in Sec. 4. This property is the basis of the asymptotic behavior of the coupled system to be considered in Sec. 5: atmospheric temperature evolution (with oceanic feedback), random attractors, finite dimensionality and ergodicity. Finally, we summarize these results in Sec. 6.

2. A Coupled Atmosphere-Ocean Model

We consider a zonally averaged, coupled atmosphere-ocean model on the meridional, latitude-depth (y, z) -plane as used by various authors [27, 31, 3, 10, 9]. It is composed of a one-dimensional stochastic energy balance model proposed by North and Cahalan [19], for the latitudinal atmosphere surface temperature $\Theta(y, t)$ on domain $0 < y < 1$, together with the Boussinesq equations for ocean dynamics in terms of vorticity $q(y, z, t)$, and transport equations for the oceanic salinity $S(y, z, t)$ and the oceanic temperature $T(y, z, t)$ on the domain $D = \{(y, z) : 0 \leq y, z \leq 1\}$:

$$\begin{aligned} \Theta_t &= \Theta_{yy} - (a + \Theta) + S_a(y) - b(y)(S_o(y) + \Theta - T(y, 1)) + \dot{w}, \\ q_t + J(q, \psi) &= \text{Pr}\Delta q + \text{Pr} \cdot \text{Ra}(\partial_y T - \partial_y S_y), \\ T_t + J(T, \psi) &= \Delta T, \\ S_t + J(S, \psi) &= \Delta S, \end{aligned} \tag{1}$$

where

$$q(y, z, t) = -\Delta\psi$$

is the vorticity, a is a positive constant parametrizing the effect of the earth's longwave radiative cooling, $S_a(y)$ and $S_o(y)$ are empirical functions representing the latitudinal dependence of the shortwave solar radiation, $b(y)$ is the latitudinal fraction of the earth covered by the ocean basin, Pr is the Prandtl number and Ra is the Rayleigh number. The first equation is the energy balance model proposed by North and Cahalan [19]. The fluctuating forcing $\dot{w}(y, t)$ may arise from, for example, eddy transport fluctuation, stormy bursts of latent heat, and flickering cloudiness variables. This forcing term is usually of a shorter time scale than the response time scale of the large scale oceanic thermohaline circulation. So we neglect the autocorrelation time of this fluctuating process as in [19]. We thus assume that the noise is white in time. The random white-in-time noise $\dot{w}(y, t)$ is described as the generalized time derivative of a Wiener process $w(y, t)$ with mean zero and covariance operator Q . Moreover, $J(g, h) = g_x h_y - g_y h_x$ is the Jacobian operator

and $\Delta = \partial_{yy} + \partial_{zz}$ is the Laplacian operator. All these equations are in non-dimensionalized forms.

Note that the Laplacian operator Δ in the temperature and salinity transport equations is presumably $\partial_{yy} + \frac{\kappa_H}{\kappa_V} \delta^2 \partial_{zz}$ with δ being the aspect ratio, and κ_H, κ_V the horizontal and vertical diffusivities of heat/salt, respectively. However, our energy-type estimates and the results below will not be essentially affected by taking a homogenized Laplacian operator $\Delta = \partial_{yy} + \partial_{zz}$. All our results would be true for this modified Laplacian. The effect of the rotation is parametrized in the magnitude of the viscosity and diffusivity terms as discussed in [30].

The no-flux boundary condition is taken for the atmosphere temperature $\Theta(y, t)$

$$\Theta_y(0, t) = \Theta_y(1, t) = 0.$$

The fluid boundary condition is no normal flow and free-slip on the whole boundary

$$\psi = 0, \quad q = 0.$$

The flux boundary conditions are assumed for the ocean temperature T and salinity S .

At top $z = 1$, the fluxes are specified as:

$$\partial_z T(y, 1) = S_o(y) + \Theta(y) - T(y, 1), \quad \partial_z S(y, 1) = F(y), \quad (2)$$

with $F(y)$ being the given freshwater flux.

At bottom $z = 0$:

$$\partial_z T = \partial_z S = 0.$$

On the lateral boundary $y \in \{0, 1\}$:

$$\partial_y T = \partial_y S = 0.$$

The stochastic partial differential equation for air temperature Θ in (1) is only defined on the air–sea interface ($0 \leq y \leq 1$) and it may be regarded as a dynamical boundary condition. The boundary condition (2) involves a coupling between the atmospheric and oceanic temperature at the air–sea interface.

The deterministic version of this model was studied in [14]. Now we look at the well-posedness of this coupled atmosphere-ocean model and then investigate its random dynamics.

3. Well-Posedness

In this section we will show that (1) defines a well-posed model. In particular, we can show that (1) has a unique solution. Without such a property it would not be possible to make predictions from the model numerical simulations or investigate the stability behavior.

Now we are going to reformulate the model such that appropriate tools of the theory of random dynamical systems can be applied to analyse the coupled

atmosphere-ocean model under a random wind forcing. For the following we need some tools from the theory of partial differential equations.

Let $W_2^1(D)$ be the Sobolev space of functions on D with first generalized derivative in $L_2(D)$, the function space of square integrable functions on D with norm and inner product

$$\|u\|_{L_2} = \left(\int_D |u(x)|^2 dD \right)^{1/2}, \quad (u, v)_{L_2} = \int_D u(x)v(x)dD, \quad u, v \in L_2(D).$$

The space $W_2^1(D)$ is equipped with the norm

$$\|u\|_{W_2^1} = \|u\|_{L_2} + \|\partial_y u\|_{L_2} + \|\partial_z u\|_{L_2}.$$

Motivated by the zero-boundary conditions of q we also introduce the space $\overset{\circ}{W}_2^1(D)$ which contains roughly speaking functions which are zero on the boundary ∂D of D . This space can be equipped with the norm

$$\|u\|_{\overset{\circ}{W}_2^1(D)} = \|\partial_y u\|_{L_2} + \|\partial_z u\|_{L_2}. \tag{3}$$

Similarly, we can define function spaces on the interval $(0, 1)$ denoted by $L_2(0, 1)$ and $W_2^1(0, 1)$.

Another Sobolev space is given by $\dot{W}_1^2(D)$ which is a subspace of $W_2^1(D)$ consisting of functions u such that $\int_D u dD = 0$. A norm equivalent to the W_2^1 -norm on $\dot{W}_1^2(D)$ is given by the right-hand side of (3). For functions in $L_2(D)$ having this property we will write $\dot{L}_2(D)$.

Since D has a Lipschitz continuous boundary ∂D there exists a *continuous trace operator*:

$$\gamma_{\partial D} : W_2^1(D) \rightarrow H^{1/2}(\partial D).$$

Here $H^{1/2}(\partial D)$ is a boundary space, see Adams [1] or below. Similarly, we can introduce trace operators that map onto a part of the boundary of ∂D for instance for the subset $\{(y, z) \in \bar{D} | z = 1\}$ of \bar{D} . For this mapping we will write

$$\gamma_{z=1} : W_2^1(D) \rightarrow H^{1/2}(0, 1). \tag{4}$$

The adjoint operator

$$\gamma_{z=1}^* : (H^{1/2}(0, 1))' \rightarrow (W_2^1(D))'$$

is also continuous. Note that $'$ denotes the dual space for a given Banach space.

Our intention is now to formulate the problem (1) with the nonhomogeneous boundary conditions in a weak form. For convenience, we introduce the vector notation for unknown geophysical quantities

$$u = (\Theta, q, T, S). \tag{5}$$

We now take the linear differential operator from (1):

$$\mathcal{A}u = \begin{pmatrix} -\partial_{yy}^2 \Theta + (1 + b(y))\Theta \\ -\text{Pr}\Delta q \\ -\Delta T \\ -\Delta S \end{pmatrix}.$$

Recall that the function $0 < b(y) < 10$. \mathcal{A} is defined on functions that are sufficiently smooth. We also have the following boundary conditions from Sec. 2

$$\begin{aligned} \partial_y \Theta(0) &= \partial_y \Theta(1) = 0, \\ \psi|_{\partial D} &= 0, \quad q|_{\partial D} = 0, \\ \partial_z T(y, 1) &= S_o(y) + \Theta(y) - T(y, 1), \quad \partial_z S(y, 1) = F(y), \\ \partial_z T(y, 0) &= \partial_z S(y, 0) = 0, \\ \partial_y T(0, z) &= \partial_y S(0, z) = 0, \quad \partial_y T(1, z) = \partial_z S(1, z) = 0. \end{aligned}$$

We will assume that S_o, S_a and $F \in L_2(0, 1)$. Note that

$$\frac{d}{dt} \int_{\Omega} S dy dz = \int_0^1 F(y) dy = \text{const.}$$

It is reasonable (see [9]) to assume that

$$\int_0^1 F(y) dy = 0, \tag{6}$$

and thus $\int_D S dy dz$ is constant in time and we may assume that it is zero:

$$\int_D S dy dz = 0.$$

Thus we have the usual Poincaré inequality for S . Unfortunately, this is not the situation for T . However, we can derive the following Poincaré inequality

$$\|T\|^2 \leq 2\|\gamma_{z=1} T\|_{L^2}^2 + 4\|\nabla T\|^2, \tag{7}$$

as in Temam [29], p. 52.

We introduce the phase space for our geophysical quantities $H = L_2(0, 1) \times L_2(D) \times L_2(D) \times \dot{L}_2(D)$ with the usual L_2 inner product and $V = W_2^1(0, 1) \times \dot{W}_2^1(D) \times W_2^1(D) \times \dot{W}_2^1(D)$. For another sufficiently smooth functions $v = (\bar{\Theta}, \bar{q}, \bar{T}, \bar{S})$, we can calculate via integration by parts

$$\begin{aligned} (\mathcal{A}u, v)_H &= \int_{(0,1)} \partial_y \Theta \partial_y \bar{\Theta} dy + \int_0^1 (1 + b) \Theta \bar{\Theta} dy \\ &+ \text{Pr} \int_D \nabla q \cdot \nabla \bar{q} dD \\ &+ \int_D \nabla T \cdot \nabla \bar{T} dD - \int_0^1 (S_o(y) + \Theta(y) - \gamma_{z=1} T(y, z)) (\gamma_{z=1} \bar{T}(y, z)) dy \\ &+ \int_D \nabla S \cdot \nabla \bar{S} dD - \int_0^1 F(y) (\gamma_{z=1} \bar{S}(y, z)) dy. \end{aligned} \tag{8}$$

Hence on the space V we can introduce a bilinear form $\tilde{a}(\cdot, \cdot)$ which is continuous, symmetric and positive

$$\begin{aligned} \tilde{a}(u, v) &= \int_{(0,1)} \partial_y \Theta \cdot \partial_y \bar{\Theta} dy + \int_0^1 (1+b)\Theta \bar{\Theta} dy + \Pr \int_D \nabla q \cdot \nabla \bar{q} dD \\ &\quad + \int_D \nabla T \cdot \nabla \bar{T} dD + c_0 \int_0^1 \gamma_{z=1} T \gamma_{z=1} \bar{T} dy + \int_D \nabla S \cdot \nabla \bar{S} dD \end{aligned}$$

for some sufficiently small $c_0 > 0$. The other terms from (8) will be considered separately. This bilinear form defines a unique linear continuous operator $A : V \rightarrow V'$ such that $\langle Au, v \rangle = \tilde{a}(u, v)$. We can see that the bilinear form $\tilde{a}(u, v)$ is positive, using the Poincaré inequality (7). According to (8), we now introduce the nonlinear operator $F(u) := F_1(u) + F_2(u)$ where

$$F_1(u)[y, z] = \begin{pmatrix} 0 \\ -J(q, \psi) \\ -J(T, \psi) \\ -J(S, \psi) \end{pmatrix} [y, z]$$

and

$$F_2(u)[y, z] = \begin{pmatrix} -a + S_a(y) - b(y)(S_o(y) - \gamma_{z=1}T) \\ \Pr \text{Ra}(\partial_y T - \partial_y S) \\ \gamma_{z=1}^*(S_o(y) + \Theta - \gamma_{z=1}T) \\ \gamma_{z=1}^* F(y) \end{pmatrix} [y, z].$$

Lemma 3.1. *The operator $F_1 : V \rightarrow H$ is continuous. In particular, we have*

$$\langle F_1(u), u \rangle = 0.$$

Proof. We have a constant $c_1 > 0$ such that

$$\|\psi\|_{W_2^3(D)} \leq c_1 \|q\|_{W_2^1(D)} \tag{9}$$

for any $q \in W_2^1(D)$ which follows straightforwardly by regularity properties of a linear elliptic boundary problem. Note that W_2^3 is a Sobolev space with respect to the third derivatives. Hence we get:

$$\begin{aligned} \|J(T, \psi)\|_{L_2} &\leq \sup_{(y,z) \in D} (|\partial_y \psi(y, z)| + |\partial_z \psi(y, z)|) \\ &\quad \times \left(\int_D |\partial_y T(y, z)| + |\partial_z T(y, z)| dD \right). \end{aligned}$$

The second factor on the right-hand side is bounded by

$$\left(\int_D |\partial_y T(y, z)|^2 dD \right)^{1/2} + \left(\int_D |\partial_z T(y, z)|^2 dD \right)^{1/2} \leq \|u\|_V.$$

On account of the Sobolev embedding lemma, we have some positive constants c_2, c_3 such that

$$\sup_{(y,z) \in D} (|\partial_y \psi(y, z)| + |\partial_z \psi(y, z)|) \leq c_2 \|\nabla \psi\|_{W_2^2(D)} \leq c_3 \|q\|_{W_2^1(D)} \leq c_3 \|u\|_V.$$

Hence we have a positive constant c_4 such that

$$\|J(T, \psi)\|_{L_2} \leq c_4 \|u\|_V^2$$

for $u \in V$. Similarly, we can treat the other terms containing J .

We now show that

$$\langle J(T, \psi), T \rangle = 0.$$

For the other terms containing J we get a similar property. We obtain via integration by parts

$$\begin{aligned} & \int_D \partial_y T \partial_z \psi T dD - \int_D \partial_z T \partial_y \psi T dD \\ &= - \int_D \partial_{yz}^2 T \psi T dD + \int_D \partial_{zy}^2 T \psi T dD - \int_D \partial_y T \psi \partial_z T dD + \int_D \partial_z T \psi \partial_y T dD \\ & \quad + \int_{(0,1)} \partial_y T \psi T|_{z=0}^{z=1} dy - \int_{(0,1)} \partial_z T \psi T|_{y=0}^{y=1} dz = 0 \end{aligned}$$

because ψ is zero on the boundary ∂D . This relation is true for a set of sufficiently smooth functions ψ, T which are dense in $\overset{\circ}{W}_2^1(D) \times W_2^1(D)$. By the continuity of F_1 , as just shown in Lemma 3.1, we can extend this property to $W_2^1(D) \times W_2^1(D)$. □

Lemma 3.2. *The following estimate holds*

$$\|F_2(u)\|_{V'} \leq c_5 \|u\|_V + c_6$$

for some positive constants c_5, c_6 .

Proof. Let $\zeta \in W_2^1(D)$. Since $\gamma_{z=1} T \in H^{1/2}(0, 1) \subset H^{-1/2}(0, 1)$, we have $\gamma_{z=1}^* \gamma_{z=1} T \in (W_1^2(D))'$ and

$$|\langle \gamma_{z=1}^* \gamma_{z=1} T, \zeta \rangle| = |(\gamma_{z=1} T, \gamma_{z=1} \zeta)| \leq c_7 \|T\|_{W_2^1} \|\zeta\|_{W_2^1}$$

for $c_7 > 0$, which immediately gives the first part of the above inequality. The other parts can be treated similarly. □

After this preparation, we are able to write our problem as a stochastic evolution equation. An introduction into the theory of stochastic differential equations can be found in Zabczyk [32].

Let \dot{w} be a noise on $L_2(D)$ with finite energy given by the covariance operator Q of the Wiener process $w(t)$ which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the vector

$$W = (w, 0, 0, 0),$$

we rewrite the coupled atmosphere-ocean system (1) as a stochastic differential equation on V' :

$$\frac{du}{dt} + Au = F(u) + \dot{W}, \quad u(0) = u_0 \in H, \tag{10}$$

where \dot{w} is a white noise as the generalized temporal derivative of a Wiener process w with continuous trajectories on \mathbb{R} and with values in $L_2(0, 1)$. Sufficient for this regularity is that the trace of the covariance is finite with respect to the space $L_2(0, 1)$: $\text{tr}_{L_2} Q < \infty$. In particular, we can choose the canonical probability space where the set of elementary events Ω consists of the paths of w and the probability measure \mathbb{P} is the Wiener measure with respect to covariance Q .

In the following, we need a stationary Ornstein–Uhlenbeck process solving the linear stochastic equation on $(0, 1)$

$$\frac{dz}{dt} + A_1 z = \dot{w}, \tag{11}$$

where $A_1 = -\partial_{yy} + (1 + b(y))$ is the linear operator with the homogeneous Neumann boundary condition at $y = 0$ and $y = 1$.

Lemma 3.3. *Suppose that the covariance Q has a finite trace: $\text{tr}_{L_2} Q < \infty$. Then (11) has a unique stationary solution generated by*

$$(t, \omega) \rightarrow z(\theta_t \omega).$$

Moreover, $Z(\omega) = (z(\omega), 0, 0, 0)$ is a random variable in V .

For the proof we refer to Da Prato and Zabczyk [20], Chapter 5, or Chueshov and Scheutzow [6].

For our calculations it will be appropriate to transform (10) into a differential equation without white noise but with random coefficients. We set

$$v := u - Z. \tag{12}$$

Thus we obtain a random differential equation in V'

$$\frac{dv}{dt} + Av = F_1(v) + F_2(v + Z(\theta_t \omega)), \quad v(0) = v_0 \in H. \tag{13}$$

Equivalently, we can formulate Eq. (13) using test functions

$$\frac{d}{dt}(v(t), \zeta) + a(v(t), \zeta) = (F_1(v(t)), \zeta) + (F_2(v(t) + Z(\theta_t \omega)), \zeta) \quad \text{for all } \zeta \in V.$$

We have obtained a differential equation without white noise but with random coefficients. Such a differential equation can be treated sample-wise for *any* sample ω . Hence it is simpler to consider (13) than to study the stochastic differential equation (10) directly. We are looking for solutions in

$$v \in C([0, \tau]; H) \cap L^2(0, \tau; V),$$

for all $\tau > 0$. If we can solve this equation, then $u := v + Z$ defines a solution version of (10). For the well-posedness of the problem we now have the following result.

Theorem 3.4. (Well-posedness) *For any time $\tau > 0$, there exists a unique solution of (13) in $C([0, \tau]; H) \cap L_2(0, \tau; V)$. In particular, the solution mapping*

$$\mathbb{R}^+ \times \Omega \times H \ni (t, \omega, v_0) \rightarrow v(t) \in H$$

is measurable in its arguments and the solution mapping $H \ni v_0 \rightarrow v(t) \in H$ is continuous.

Proof. By the properties of A and F_1 (see Lemma 3.1), the random differential equation (13) is essentially similar to the two-dimensional Navier–Stokes equation. Note that F_2 is only an affine mapping. Hence we have existence and uniqueness and the above regularity assertions. □

On account of the transformation (12), we find that (10) also has a unique solution. Since the solution mapping

$$\mathbb{R}^+ \times \Omega \times H \ni (t, \omega, v_0) \rightarrow v(t, \omega, v_0) =: \varphi(t, \omega, v_0) \in H$$

is well defined, we can introduce a random dynamical system. On Ω we can define a shift operator θ_t on the paths of the Wiener process that pushes our noise:

$$w(\cdot, \theta_t \omega) = w(\cdot + t, \omega) - w(t, \omega) \quad \text{for } t \in \mathbb{R}$$

which is called the *Wiener shift*. Then $\{\theta_t\}_{t \in \mathbb{R}}$ forms a flow which is ergodic for the probability measure \mathbb{P} . The properties of the solution mapping cause the following relations

$$\varphi(t + \tau, \omega, u) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, u)) \quad \text{for } t, \tau \geq 0$$

$$\varphi(0, \omega, u) = u$$

for any $\omega \in \Omega$ and $u \in H$. This property is called the cocycle property of φ which is important to study the dynamics of random systems. It is a generalization of the semigroup property. The cocycle φ together with the flow θ forms a *random dynamical system*.

4. Dissipativity

In this section we are going to show that the coupled atmosphere-ocean system (1) is dissipative, in the sense that it has an absorbing (random) set. This definition has been used for deterministic systems [29]. This means that the solution vector v is contained in a particular region of the phase space H after a sufficiently long time. Dissipativity will be very important for understanding the asymptotic dynamics of the system. This dissipativity will give us estimate of the atmospheric temperature evolution under oceanic feedback. Dynamical properties that follow from this dissipativity will be considered in the next section. In particular, we will show that

the coupled atmosphere-ocean system has a random attractor, has finite degree of freedom, and is ergodic under suitable conditions.

We introduce the spaces

$$\begin{aligned} \tilde{H} &= L_2(0, 1) \times L_2(D) \times \dot{L}_2(D), \\ \tilde{V} &= W_2^1(0, 1) \times W_2^1(D) \times \dot{W}_2^1(D). \end{aligned}$$

We also choose a subset of dynamical variables of our system (1).

$$\tilde{v} = (\tilde{\Theta}, T, S), \quad \tilde{\Theta} = \Theta - z. \tag{14}$$

To calculate the energy inequality for \tilde{v} , we apply the chain rule to $\|\tilde{v}\|_H^2$. We obtain by Lemma 3.1

$$\begin{aligned} &\frac{d}{dt}\|\tilde{v}\|_H^2 + 2\|\nabla\tilde{v}\|_{L_2}^2 + 2\|b^{\frac{1}{2}}\tilde{\Theta}\|_{L_2}^2 + 2\|\tilde{\Theta}\|_{L_2}^2 \\ &= -2(a, \tilde{\Theta})_{L_2} + 2(S_a, \tilde{\Theta})_{L_2} - 2(bS_o, \tilde{\Theta})_{L_2} + 2(b\gamma_{z=1}T, \tilde{\Theta})_{L_2} \\ &\quad + 2\langle\gamma_{z=1}^*S_o, T\rangle + 2\langle\gamma_{z=1}^*\tilde{\Theta}, T\rangle + 2\langle\gamma_{z=1}^*z(\theta_t\omega), T\rangle - 2\langle\gamma_{z=1}^*\gamma_{z=1}T, T\rangle \\ &\quad + 2\langle\gamma_{z=1}^*F, S\rangle. \end{aligned} \tag{15}$$

Here and in the following we stress that $0 < b(y) < 1$. The expression $\nabla\tilde{v}$ is defined by $(\partial_y\tilde{\Theta}, \nabla_{y,z}T, \nabla_{y,z}S)$. We can now estimate the terms on the right-hand side. We have the following estimate for the second line of (15) by the Cauchy–Schwarz inequality

$$\begin{aligned} &4a^2 + \frac{1}{4}\|\tilde{\Theta}\|_{L_2}^2 + 4\|S_a\|_{L_2}^2 + \frac{1}{4}\|\tilde{\Theta}\|_{L_2}^2 \\ &\quad + 2\|b^{1/2}S_o\|_{L_2}^2 + \frac{1}{2}\|b^{\frac{1}{2}}\tilde{\Theta}\|_{L_2}^2 + \frac{2}{3}\|b^{1/2}\|_{L_\infty}\|\gamma_{z=1}T\|_{L_2}^2 + \frac{3}{2}\|b^{\frac{1}{2}}\tilde{\Theta}\|_{L_2}^2. \end{aligned}$$

For the next line we obtain the estimate

$$\begin{aligned} &6\|S_o\|_{L_2}^2 + \frac{1}{6}\|\gamma_{z=1}T\|_{L_2}^2 \\ &\quad + \|\tilde{\Theta}\|_{L_2}^2 + \|\gamma_{z=1}T\|_{L_2}^2 + 6\|z(\theta_t\omega)\|_{L_2}^2 + \frac{1}{6}\|\gamma_{z=1}T\|_{L_2}^2 - 2\|\gamma_{z=1}T\|_{L_2}^2. \end{aligned}$$

Now we can estimate the last line of (15). For any $\varepsilon > 0$ we can find an $c_8(\varepsilon) > 0$ such that

$$\varepsilon\|S\|_{W_2^1}^2 + c_8(\varepsilon)\|F\|_{L_2}^2.$$

Here we also applied the trace theorem $\|\gamma_{z=1}S\|_{H^{1/2}} \leq c_9\|S\|_{W_2^1}$. Adding all terms containing $\|\gamma_{z=1}T\|_{L_2}^2$, we see that the sum is negative.

Collecting all these estimates, we have

$$\begin{aligned} &\frac{d}{dt}\|\tilde{v}\|_H^2 + 2\|\nabla\tilde{v}\|_{L_2}^2 + \frac{2}{3}(1 - \|b^{1/2}\|_{L_\infty})\|\gamma_{z=1}T\|_{L_2}^2 + \frac{1}{2}\|\tilde{\Theta}\|_{L_2}^2 \\ &\leq 4a^2 + 4\|S_a\|_{L_2}^2 + 2\|b^{1/2}S_o\|_{L_2}^2 + 6\|S_o\|_{L_2}^2 + 6\|z(\theta_t\omega)\|_{L_2}^2 + c_8(\varepsilon)\|F\|_{L_2}^2. \end{aligned}$$

By using the Poincaré inequality for $S \in \dot{W}_2^1(D)$, (7), and choosing ε small enough, we conclude that there is a positive dissipativity constant α such that

$$\frac{d}{dt} \|\tilde{v}\|_{\tilde{H}}^2 + \alpha(\|\tilde{v}\|_{\tilde{H}}^2 + \|\nabla \tilde{v}\|_{L_2}^2) \leq c_{10} + 6\|z(\theta_t \omega)\|_{L_2}^2, \tag{16}$$

where α, c_{10} is determined by physical data $a^2, \|S_a\|_{L_2}, \|S_o\|_{L_2}, \|F\|_{L_2}$ and $\|b\|_{L^\infty}$.

By the Gronwall inequality, we finally conclude that

$$\|\tilde{v}\|_{\tilde{H}}^2 \leq \|\tilde{v}(0)\|_{\tilde{H}}^2 e^{-\alpha t} + \frac{c_{10}}{\alpha} + 6e^{-\alpha t} \int_0^t \|z(\theta_s \omega)\|_{L_2}^2 e^{\alpha s} ds. \tag{17}$$

We now show the dissipativity of \tilde{v} and v . Roughly speaking dissipativity means that all trajectories of the system move to a bounded set in the phase space. For a random system we have the following version of dissipativity.

Definition 4.1. A random set $B = \{B(\omega)\}_{\omega \in \Omega}$ consisting of closed bounded sets $B(\omega)$ is called absorbing for a random dynamical system φ if we have for any random set $D = \{D(\omega)\}_{\omega \in \Omega}$, $D(\omega) \in H$ bounded, such that $t \rightarrow \sup_{y \in D(\theta_t \omega)} \|y\|_H$ has a subexponential growth for $t \rightarrow \pm\infty$

$$\begin{aligned} \varphi(t, \omega, D(\omega)) &\subset B(\theta_t \omega) \quad \text{for } t \geq t_0(D, \omega) \\ \varphi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)) &\subset B(\omega) \quad \text{for } t \geq t_0(D, \omega). \end{aligned} \tag{18}$$

B is called forward invariant if

$$\varphi(t, \omega, u_0) \in B(\theta_t \omega) \quad \text{if } u_0 \in B(\omega) \quad \text{for } t \geq 0.$$

Although \tilde{v} is not a random dynamical system in the strong sense we can also show dissipativity in the sense of the above definition.

Lemma 4.2. Let $\tilde{\varphi}(t, \omega, v_0) \in \tilde{H}$ for $v_0 \in H$ be defined in (10). Then the closed ball $B(0, R_1(\omega))$ with radius

$$R_1(\omega) = 2 \int_{-\infty}^0 e^{\alpha \tau} (c_{10} + 6\|z(\theta_\tau \omega)\|_{L_2}^2) d\tau$$

is forward invariant and absorbing.

The proof of this lemma follows by integration of (16).

It remains to prove the dissipativity of the dynamical system φ . To this end we obtain from the second equation of (1):

$$\begin{aligned} \frac{d}{dt} \|q\|_{L_2}^2 + c_{11}^2 \Pr \|q\|_{L_2}^2 &\leq \Pr \text{Ra}^2 \|\nabla \tilde{v}\|_{L_2}^2 \\ &\leq \frac{\Pr \text{Ra}^2 c_{10}}{\alpha} + 6 \frac{\Pr \text{Ra}^2}{\alpha} \|z(\theta_t \omega)\|_{L_2}^2 - \frac{\Pr \text{Ra}^2}{\alpha} \frac{d}{dt} \|\tilde{v}\|_{\tilde{H}}^2 \end{aligned}$$

with some embedding constant c_{11} in the Poincaré inequality $\|q\|_{L_2} \leq c_{11} \|\nabla q\|_{L_2}$ for $q \in \dot{W}_2^1(D)$. Note that q satisfies homogeneous Dirichlet boundary conditions. Hence the variation of constants formula allows us to estimate:

$$\begin{aligned} \|q(t, \omega, u_0)\|_{L_2}^2 &\leq \|v_0\|_H^2 e^{-c_{11}^2 \text{Pr} t} + \int_0^t \left(\frac{\text{Pr Ra}^2 c_{10}}{\alpha} + 6 \frac{\text{Pr Ra}^2}{\alpha} \|z(\theta_s \omega)\|_{L_2}^2 \right. \\ &\quad \left. - \frac{\text{Pr Ra}^2}{\alpha} \frac{d}{ds} \|\tilde{v}(s)\|_H^2 \right) e^{-c_{11}^2 \text{Pr}(t-s)} ds. \end{aligned} \tag{19}$$

Now we apply the integration by parts:

$$\begin{aligned} - \int_0^t \frac{d}{ds} \|\tilde{v}(s)\|_H^2 e^{c_{11}^2 \text{Pr} s} ds &= \int_0^t c_{11}^2 \text{Pr} \|\tilde{v}(s)\|_{L_2}^2 e^{c_{11}^2 \text{Pr} s} ds \\ &\quad - e^{c_{11}^2 \text{Pr} t} \|\tilde{v}(t)\|_{L_2}^2 + \|\tilde{v}(0)\|_{L_2}^2. \end{aligned}$$

Substituting this equation into (19), we obtain

$$\begin{aligned} \|q(t, \omega, u)\|_{L_2}^2 &\leq \|v_0\|_H^2 e^{-c_{11}^2 \text{Pr} t} \\ &\quad + \int_0^t \left(\frac{\text{Pr Ra}^2}{\alpha} c_{10} + 6 \frac{\text{Pr Ra}^2}{\alpha} \|z(\theta_s \omega)\|_{L_2}^2 \right) e^{-c_{11}^2 \text{Pr}(t-s)} ds \\ &\quad + \int_0^t \left(\frac{c_{11}^2 \text{Ra}^2 \text{Pr}^2}{\alpha} \|\tilde{v}(s)\|_H^2 \right) e^{-c_{11}^2 \text{Pr}(t-s)} ds \\ &\quad + e^{-c_{11}^2 \text{Pr} t} \frac{\text{Pr Ra}^2}{\alpha} \|\tilde{v}(0)\|_H^2. \end{aligned} \tag{20}$$

Note that $\|\tilde{v}(t)\|_H^2$ is bounded by

$$\|\tilde{v}_0\|_H^2 e^{-\alpha t} + R_1(\theta_t \omega)$$

which follows from (17). To construct the radius of the absorbing set we have to replace ω by $\theta_{-t}\omega$. Suppose that $t \rightarrow \|v_0(\theta_{-t}\omega)\|_H^2$ grows not faster than subexponential. Then we have that

$$\lim_{t \rightarrow \infty} \|v_0(\omega)\|_H^2 e^{-\alpha t} = 0, \quad \lim_{t \rightarrow \infty} \|v_0(\theta_{-t}\omega)\|_H^2 e^{-\alpha t} = 0.$$

Hence we can conclude

$$\lim_{t \rightarrow \infty} \int_0^t \frac{c_{11}^2 \text{Ra}^2 \text{Pr}^2}{\alpha} \|v_0(\omega)\|_H^2 e^{-\alpha t} e^{-c_{11}^2 \text{Pr}(t-s)} ds = 0$$

and

$$\lim_{t \rightarrow \infty} \int_0^t c_{11}^2 \frac{\text{Ra}^2 \text{Pr}^2}{\alpha} \|v_0(\theta_{-t}\omega)\|_H^2 e^{-\alpha t} e^{-c_{11}^2 \text{Pr}(t-s)} ds = 0.$$

We also note that

$$\begin{aligned} &\lim_{t \rightarrow \infty} \int_0^t \left(\frac{\text{Pr Ra}^2 c_{10}}{\alpha} + 6 \frac{\text{Pr Ra}^2}{\alpha} \|z(\theta_{t-s}\omega)\|_{L_2}^2 + \frac{c_{11}^2 \text{Ra}^2 \text{Pr}^2}{\alpha} R_1(\theta_{s-t}\omega) \right) e^{-c_{11}^2 \text{Pr}(t-s)} ds \\ &= \lim_{t \rightarrow \infty} \int_{-t}^0 \left(\frac{\text{Pr Ra}^2 c_{10}}{\alpha} + 6 \frac{\text{Pr Ra}^2}{\alpha} \|z(\theta_s \omega)\|_{L_2}^2 + \frac{c_{11}^2 \text{Ra}^2 \text{Pr}^2}{\alpha} R_1(\theta_s \omega) \right) e^{c_{11}^2 \text{Pr} s} ds \\ &=: \frac{R_2(\omega)}{2} < \infty. \end{aligned}$$

The finiteness of this limit follows because the growth of $t \rightarrow R_1(\theta_t \omega)$ is subexponential. Since all other terms are coupled with exponentially decreasing factors we have found:

Lemma 4.3. *Suppose that the assumptions of Lemma 4.2 are satisfied. Then the random set $\{B(\omega)\}_{\omega \in \Omega}$ given by closed balls $B(0, R(\omega))$ in H with center zero and radius $R(\omega) := R_1(\omega) + R_2(\omega)$ is an absorbing and forward invariant set denoted by $B(\omega)$ for the random dynamical system generated by (13).*

For the applications in the next section we need that the elements which are contained in the absorbing set satisfy a particular regularity. To this end we introduce the function space

$$\mathcal{H}^s := \{u \in H : \|u\|_s^2 := \|A^{s/2}u\|_H^2 < \infty\},$$

where $s \in \mathbb{R}$. The operator A^s is the s th power of the positive and symmetric operator A . Note that these spaces are embedded in the Slobodeckij spaces H^s , $s > 0$. The norm of these spaces is denoted by $\|\cdot\|_{H^s}$. This norm can be found in Egorov and Shubin [11], p. 118. But we do not need this norm explicitly. We only mention that on \mathcal{H}^s the norm $\|\cdot\|_s$ of H^s is equivalent to the norm of \mathcal{H}^s for $0 < s$, see [17].

The reason to introduce these spaces is that the trace theorem can be formulated with respect to H^s ; see Egorov and Shubin [11], p. 120:

Theorem 4.4. *Assume that $\alpha > 1/2$. Then the trace mapping between two Sobolev spaces*

$$\gamma_{z=1} : H^\alpha(D) \rightarrow H^{\alpha-1/2}(0, 1)$$

is continuous.

This formula generalized (4) because we can take for $\alpha = 1$ and $H^1 = W_2^1(D)$ or V . Our goal is to show that $v(1, \omega, D)$ is a bounded set in \mathcal{H}^s for some $s > 0$. This property causes the complete continuity of the mapping $v(1, \omega, \cdot)$. We now derive a differential inequality for $t\|v(t)\|_s^2$. By the chain rule we have

$$\frac{d}{dt}(t\|v(t)\|_s^2) = \|v(t)\|_s^2 + t\frac{d}{dt}\|v(t)\|_s^2.$$

Note that for the embedding constant $c_{12,s}$ between \mathcal{H}^s and V

$$\int_0^t \|v\|_s^2 ds \leq c_{12,s}^2 \int_0^t \|v\|_V^2 ds \quad \text{for } s \leq 1$$

such that the left-hand side is bounded if the initial conditions v_0 are contained in a bounded set in H . The second term in the above formula can be expressed as follows:

$$\begin{aligned} t\frac{d}{dt}(A^{s/2}v, A^{s/2}v)_H &= 2t\left(\frac{d}{dt}v, A^s v\right)_H = -2t(Au, A^s v)_H + 2t(F_1(v), A^s v)_H \\ &\quad + 2t(F_2(v + Z(\theta_t \omega)), A^s v)_H. \end{aligned}$$

We have

$$(Av, A^s v)_H = \|A^{1/2+s/2}v\|_H = \|v\|_{1+s}^2.$$

If we apply some embedding theorems, see Temam [28] p. 12 we have got for a $c_{13} > 0$

$$(F_1(v), \zeta)_H \leq c_{13} \|v\|_{m_1+1} \|\psi\|_{m_2+1} \|\zeta\|_{m_3}, \quad \zeta \in H_{m_3},$$

where $m_1 + m_2 + m_3 \geq 1$ and $0 \leq m_i < 1$. Here we use that D is of dimension 2. We then have for $m_1 = 0, m_2 = s < 1$ and $m_3 = 1 - s$

$$|(F_1(v), A^s v)_H| \leq c_{13} \|v\|_V \|\psi\|_{1+s} \|v\|_{1+s}.$$

$\|\psi(t)\|_{1+s}$ is bounded by $c'_1 \|v(t)\|_H$ similar to (9) and $\|v(t)\|_{L_\infty(0,T;H)} < \infty$. To ensure that the norm of $\|\psi\|_{m_2+1}$ is well-defined we set $\psi = (0, \psi, 0, 0)$. Hence we have for any $\varepsilon > 0$ a constant $c_{14}(\varepsilon)$:

$$(F_1(v(t)), A^s v(t))_H \leq c_{14}(\varepsilon) \|v\|_{L_\infty(0,T;H)}^2 \|v(t)\|_V^2 + \varepsilon \|v(t)\|_{1+s}^2,$$

where ε is chosen sufficiently small.

To obtain an estimate for the expression $\langle F_2(v - Z), A^s v \rangle$ we only consider the expressions $\langle \gamma_{z=1}^* S_o, A^s v \rangle$ and $\langle \gamma_{z=1}^* \gamma_{z=1} T, A^s v \rangle$. The other term can be treated similarly. Suppose that $S \in L_2(0, 1)$. We interpret $\gamma^* S_o$ as $(0, 0, 0, \gamma^* S_o)$. Then we have for any $\varepsilon > 0$ a $c_{15}(\varepsilon), c_{16}(\varepsilon) > 0$ such that for $0 < s < 1/2$

$$\begin{aligned} \langle \gamma_{z=1}^* S_o, A^s v \rangle &\leq \varepsilon \|A^{\frac{1}{2}+\frac{s}{2}}v\|_H^2 + c_{15}(\varepsilon) \|A^{-\frac{1}{2}+\frac{s}{2}}\gamma_{z=1}^* S_o\|_H^2 \\ &= \varepsilon \|v\|_{1+s}^2 + c_{15}(\varepsilon) \|\gamma_{z=1}^* S_o\|_{-1+s}^2 \\ &\leq \varepsilon \|v\|_{1+s}^2 + c_{16}(\varepsilon) \|S_o\|_{H^{s-1/2}}^2 \leq \varepsilon \|v\|_{1+s}^2 + c_{16}(\varepsilon) \|S_o\|_{L_2(0,1)}^2. \end{aligned}$$

Since by the trace theorem we have $\gamma_{z=1}^* : H^{-\frac{1}{2}+s}(0, 1) \rightarrow H^{-1+s}, 1 - s > 1/2$. For $\alpha < 0$ the space H^α denotes the dual space of $H^{-\alpha}$.

For $\langle \gamma_{z=1}^* \gamma_{z=1} T, A^s v \rangle$ we have for positive constants $c_{17}-c_{19}$, a sufficiently small $\varepsilon > 0, c_{20}(\varepsilon)$ and a sufficiently small $\varepsilon' > 0$

$$\begin{aligned} \langle \gamma_{z=1}^* \gamma_{z=1} T, A^s v \rangle &= (\gamma_{z=1} T, \gamma_{z=1} A^s v)_{L_2} \\ &\leq \|\gamma_{z=1} T\|_{L_2} \|\gamma_{z=1} A^s v\|_{L_2} \\ &\leq c_{17} \|v\|_V \|\gamma_{z=1} A^s v\|_{L_2} \\ &\leq c_{18} \|v\|_V \|A^{\frac{1}{4}+\varepsilon'} A^s v\|_H \leq c_{19} \|v\|_V \|A^{\frac{1}{2}+\frac{s}{2}}v\|_H \leq c_{19} \|v\|_V \|v\|_{1+s} \\ &\leq c_{20}(\varepsilon) \|v\|_V^2 + \varepsilon \|v\|_{1+s}^2. \end{aligned}$$

For $0 < s < 1/4$ we have $A^s v \in H^{1-2s}$ for $v \in V$ such that $\gamma_{z=1} A^s v \in L_2(0, 1)$. Collecting all these estimates we obtain that $t\|v(t, \omega, v_0)\|_H$ is bounded for $t \leq T < \infty$ if v_0 is contained in a bounded set. This allows us to write down the main assertion with respect to the dissipativity of this section.

Theorem 4.5. *For the random dynamical system generated by (13), there exists a compact random set $B = \{B(\omega)\}_{\omega \in \Omega}$ which satisfies Definition 4.1.*

We define

$$B(\omega) = \overline{\varphi(1, \theta_{-1}\omega, B(0, R(\theta_{-1}\omega)))} \subset \mathcal{H}^s, \quad 0 < s < \frac{1}{4}. \tag{21}$$

In particular, \mathcal{H}^s is compactly embedded in H .

5. Random Dynamical Behavior

In this section we will apply the dissipativity result of the last section to analyze the dynamical behavior of the coupled atmosphere-ocean system (1). However, it will be enough to analyze the transformed random dynamical system generated by (10). By the transformation (12) we can take over all these qualitative properties to the system (10).

We will consider the following dynamical behavior: random climatic attractors, finite degrees of freedom, atmospheric temperature evolution under oceanic feedback, and ergodicity.

We first consider random climatic attractors. We recall the following basic concept; see, for instance, Flandoli and Schmalfuß [12].

Definition 5.1. Let φ be a random dynamical system. A random set $A = \{A(\omega)\}_{\omega \in \Omega}$ consisting of compact nonempty sets $A(\omega)$ is called random global attractor if for any random bounded set D we have for the limit in probability

$$(\mathbb{P}) \lim_{t \rightarrow \infty} \text{dist}_H(\varphi(t, \omega, D(\omega)), A(\theta_t \omega)) = 0$$

and

$$\varphi(t, \omega, A(\omega)) = A(\theta_t \omega)$$

any $t \geq 0$ and $\omega \in \Omega$.

The essential long-time behavior of a random system is captured by a random attractor. In the last section we showed that the dynamical system φ generated by (10) is dissipative which means that there exists a random set B satisfying (18). In addition, this set is compact. We now recall and adapt the following theorem from [12].

Theorem 5.2. *Let φ be a random dynamical system on the state space H which is a separable Banach space such that $x \rightarrow \varphi(t, \omega, x)$ is continuous. Suppose that B is a set ensuring the dissipativity given in Definition 4.1. In addition, B has a subexponential growth (see Definition 4.1) and is regular (compact). Then the dynamical system φ has a random attractor.*

This theorem can be applied to our random dynamical system φ generated by the stochastic differential equation (10). Indeed, all the assumptions are satisfied.

The set B is defined in Theorem 4.5. Its subexponential growth follows from $B(\omega) \subset B(0, R(\omega))$ where the radius $R(\omega)$ has been introduced in the last section. Note that φ is a *continuous* random dynamical system; see Theorem 3.4. Thus φ has a random attractor. By the transformation (12), this is also true for the original coupled atmosphere-ocean system.

Corollary 5.3. (Random attractor) *The coupled atmosphere-ocean system (1) has a random attractor.*

Dissipative systems often have finite degrees of freedom. This is reflected by the fact that the Hausdorff-dimension of the attractor is finite. This fact can be applied to fluid dynamical systems; see for instance Temam [29], p. 403 ff. A similar theory has been developed for random dynamical systems; see, for example, [7, 8, 22].

However, we will follow another approach to show that the random climatic attractor of (1) has only finitely many degrees of freedom. Namely we will use the technique of determining functionals. This technique has been introduced for deterministic systems by Foias and Prodi [13], and Ladyzhenskya [16]. See [15, 4] for more recent work. Roughly speaking, a set of determining functionals is a set of functionals (for instance, Fourier modes) such that if it is known that a dynamical system has an asymptotic stable behavior only with respect to these finitely many modes, then the complete system has an asymptotic stable behavior. For random dynamical systems we can investigate *determining functionals in probability* as in Chueshov *et al.* [5]

Definition 5.4. We call a set $\mathcal{L} = \{l_j, j = 1, \dots, N\}$ of linear continuous and linearly independent functionals on a space X continuously embedded in H (for instance $X = \mathcal{H}^s$ or V) asymptotically determining in probability if

$$(\mathbb{P}) \lim_{t \rightarrow \infty} \int_t^{t+1} \max_j |l_j(\varphi(\tau, \omega, v_1) - \varphi(\tau, \omega, v_2))|^2 d\tau = 0$$

for two initial conditions $v_1, v_2 \in H$ implies

$$(\mathbb{P}) \lim_{t \rightarrow \infty} \|\varphi(t, \omega, v_1) - \varphi(t, \omega, v_2)\|_H = 0.$$

Often the elements of set \mathcal{L} can be chosen as the projections with respect to the Fourier expansion of the solution. Since our domain D is an rectangle, we can calculate the Fourier expansion more explicitly.

In the following we need an additive embedding inequality based on qualitative difference of the spaces H and X for some set of functionals \mathcal{L}

$$\|u\|_H \leq C_{\mathcal{L}} \max_{l_i \in \mathcal{L}} |l_i(u)| + \varepsilon_{\mathcal{L}} \|u\|_X, \quad C_{\mathcal{L}} > 0. \tag{22}$$

The constant $\varepsilon_{\mathcal{L}} > 0$ describes a fundamental difference of the spaces X and H . How (22) works is described by a motivating example in Chueshov *et al.* [5]

We recall and adapt a result from [5].

Theorem 5.5. *Let $\mathcal{L} = \{l_j : j = 1, \dots, N\}$ be a set of linear continuous and linearly independent functionals on X . We assume that we have an absorbing and forward invariant set B in X such that for $\sup_{v \in B(\omega)} \|v\|_X^2$ the expression $t \rightarrow \sup_{v \in B(\theta_t \omega)} \|v\|_X^2$ is locally integrable and subexponentially growing. Suppose there exist a constant $c_{21} > 0$ and a measurable function $l \geq 0$ such that for $v_1, v_2 \in V$ we have for $F(\omega, v) = F_1(v) + F_2(v + Z(\omega))$*

$$\begin{aligned} & \langle -A(v_1 - v_2) + F(\omega, v_1) - F(\omega, v_2), v_1 - v_2 \rangle \\ & \leq -c_{21} \|v_1 - v_2\|_V^2 + l(v_1, v_2, \omega) \|v_1 - v_2\|_H^2. \end{aligned}$$

Assume that

$$\frac{1}{m} \mathbb{E} \left\{ \sup_{v_1, v_2 \in B(\omega)} \int_0^m l(\varphi(t, \omega, v_1), \varphi(t, \omega, v_2), \theta_t \omega) dt \right\} < c_{21} \varepsilon_{\mathcal{L}}^{-2}$$

for some $m > 0$. Then \mathcal{L} is a set of asymptotically determining functionals in probability for random dynamical system φ .

We set $X = \mathcal{H}^s, s \in (0, 1/4)$. In the last section we have shown that the set B , consisting of bounded sets, is forward invariant. The function l appearing in the formulation of Theorem 5.5 expresses the local Lipschitz continuity of our nonlinear operator $F(v)$. The essential part that determines F is the Jacobian operator J . A method on how to estimate the function l is in [5]. We also note that the local Lipschitz constant can be estimated in terms of the \mathcal{H}^s -norm. To estimate F_2 we have to apply the techniques introduced in Sec. 4. Note that F_2 is linear.

We can apply Theorem 5.5 to the random dynamical system generated by (10) and get the following result.

Theorem 5.6. *The random dynamical system generated by (10) has finitely many degrees of freedom. More precisely, there exists a set of linearly independent continuous functionals (on \mathcal{H}^s) which is asymptotically determining in probability.*

Due to the transformation (12), this result is also true for the coupled atmosphere-ocean system:

Corollary 5.7. (Finite degrees of freedom) *The coupled atmosphere-ocean system (1) has finitely many degrees of freedom, in the sense of having a finite set of linearly independent continuous functionals which is asymptotically determining in probability.*

The number of the elements in \mathcal{L} can be estimated explicitly as in [5].

Now we consider random fixed point and ergodicity. We can do a small modification of (1). This modification is given when we replace ΔT by $\nu \Delta T$ and ΔS by $\nu \Delta S$ where $\nu > 0$ is viscosity. Under particular assumptions about physical data in (1), we can show that the behavior of our dynamical system is laminar. For a stochastic system, this means that after a relatively short time, all trajectories starting from

different initial states show almost the same dynamical behavior. This can be seen easily if S_o, a, S_a, F are zero, there is no noise and ν is large. We will show that a laminar behavior also appears when S_o, a, S_a, F are small in some sense.

Mathematically speaking, laminar behavior means that a random dynamical system has a unique exponentially attracting random fixed point.

Definition 5.8. A random variable $v^* : \Omega \rightarrow H$ is defined to be a random fixed point for a random dynamical system if

$$\varphi(t, \omega, v^*(\omega)) = v^*(\theta_t \omega)$$

for $t \geq 0$ and $\omega \in \Omega$. A random fixed point v^* is called exponentially attracting if

$$\lim_{t \rightarrow \infty} \|\varphi(t, \omega, x) - v^*(\theta_t \omega)\|_H = 0$$

for any $x \in H$ and $\omega \in \Omega$.

Sufficient conditions for the existence of random fixed points are given in Schmalfuß [23]. We here formulate a simpler version of this theorem and it is appropriate for our system here.

Theorem 5.9. (Random fixed point theorem) *Let φ be a random dynamical system and suppose that B is a forward invariant complete set. In addition, B has a subexponential growth, see Definition 4.1. Suppose that the following contraction conditions holds:*

$$\sup_{v_1 \neq v_2 \in B(\omega)} \frac{\|\varphi(1, \omega, v_1) - \varphi(1, \omega, v_2)\|_H}{\|v_1 - v_2\|_H} \leq k(\omega), \tag{23}$$

where the expectation of $\log k$ denoted by $\mathbb{E} \log k < 0$. Then φ has a unique random fixed point in B which is exponentially attracting.

This theorem can be considered as a random version of the Banach fixed point theorem. The contraction condition is formulated in the mean for the right-hand side of (23).

Theorem 5.10. *Assume that the physical data $|a|, \|S_a\|_{L_2}, \|S_o\|_{L_2}, \|F\|_{L_2}$ and the trace of the covariance for the noise $\text{tr}_H Q$ are sufficiently small, and that the viscosity ν is sufficiently large. Then the random dynamical system generated by (10) has a unique random fixed point in B .*

Here we only give a short sketch of the proof. Let us suppose for a while that B is given by the ball $B(0, R)$ introduced in Lemma 4.3. Suppose that the data in the assumption of the lemma are small and ν is large. Then it follows that $\mathbb{E}R$ is also small. To calculate the contraction condition we have to calculate $\|\varphi(1, \omega, v_1(\omega)) - \varphi(1, \omega, v_2(\omega))\|_H^2$ for arbitrary random variables $v_1, v_2 \in B$. By Lemma 3.1 we have that

$$\langle J(q_1, \psi_1) - J(q_1, \psi_1), q_1 - q_2 \rangle \leq c_{22} \|q_1 - q_2\|_{W_2^1}^2 + c_{23} \|q_1\|_{W_2^1(D)}^2 \|q_1 - q_2\|_{L_2}^2,$$

where the constant c_{23} can be chosen sufficiently small if ν is large. On account of the fact that the other expressions allow similar estimates and that F_2 is linear we obtain:

$$\begin{aligned} & \frac{d}{dt} \|\varphi(t, \omega, v_1(\omega)) - \varphi(t, \omega, v_2(\omega))\|_H^2 \\ & \leq (-\alpha' + c_{23} \|\varphi(t, \omega, v_2(\omega))\|_V^2) \|\varphi(t, \omega, v_1(\omega)) - \varphi(t, \omega, v_2(\omega))\|_H^2 \end{aligned}$$

for some positive α' depending on ν . From this inequality and the Gronwall lemma it follows that the contraction condition (23) is satisfied if

$$\mathbb{E} \sup_{v_2 \in B(\omega)} c_{23} \int_0^1 \|\varphi(t, \omega, v_2)\|_V^2 dt < \alpha'.$$

But by the energy inequality this property is satisfied if the $\mathbb{E}R$ and $\mathbb{E}\|z\|_V^2$ is sufficiently small which follows from the assumptions.

Now let B be the random set defined in (21). Since the set B introduced in (21) is absorbing any state the fixed point v^* is contained in this B . In addition v^* attracts any state from H and not only states from B .

Corollary 5.11. (Unique random fixed point) *Assume that the physical data $|a|, \|S_a\|_{L_2}, \|S_o\|_{L_2}, \|F\|_{L_2}$ and the trace of the covariance for the noise $\text{tr}_H Q$ are sufficiently small, and that the viscosity ν is sufficiently large. Then, through the transformation (12), the original coupled atmosphere-ocean system (1) has a unique exponentially attracting random fixed point $u^*(\omega) = v^*(\omega) + Z(\omega)$, where $u = (\Theta, q, T, S)$.*

The uniqueness of this random fixed point implies *ergodicity*. We will comment on this issue at the end of this section.

By the well-posedness Theorem 3.4, we know that the stochastic differential equation (10) for the coupled atmosphere-ocean system has a unique solution. The solution is a Markov process. We can define the associated Markov operators $\mathcal{T}(t)$ for $t \geq 0$, as discussed in [25, 24]. Moreover, $\{\mathcal{T}(t)\}_{t \geq 0}$ forms a semigroup.

Let M^2 be the set of probability distributions μ with finite energy, i.e.

$$\int_H \|u\|_H^2 d\mu(u) < \infty.$$

Then the distribution of the solution $u(t)$ (at time t) of the stochastic differential equation (10) is given by

$$\mathcal{T}(t)\mu_0,$$

where the distribution μ_0 of the initial data is contained in M^2 .

We note that the expectation of the solution $\|u(t)\|_H^2$ can be expressed in terms of this distribution $\mathcal{T}(t)\mu_0$:

$$\mathbb{E}\|u(t)\|_H^2 = \int_H \|u\|_H^2 d\mathcal{T}(t)\mu_0.$$

We can derive the following energy inequality in the mean, using our earlier estimates in (17) and (20):

Theorem 5.12. *The dynamical quantity $u = (\Theta, q, T, S)$ of the coupled atmospheric-ocean system satisfy the estimate*

$$\mathbb{E}\|u(t)\|_H^2 + \alpha \mathbb{E} \int_0^t \|u(\tau)\|_V^2 d\tau \leq \mathbb{E}\|u_0\|_H^2 + t c_{24} + t \operatorname{tr}_{L_2} Q,$$

where the positive constants c_{24} and α depend on physical data $F(y)$, a , $S_a(y)$, $S_o(y)$, Pr and Ra .

By the Gronwall inequality, we further obtain the following result about the asymptotic mean-square estimate for the coupled atmosphere-ocean system.

Corollary 5.13. (Atmospheric temperature evolution under oceanic feedback) *For the expectation of the dynamical quantity $u = (\Theta, q, T, S)$ of the coupled atmospheric-ocean system, we have the asymptotic estimate*

$$\limsup_{t \rightarrow \infty} \mathbb{E}\|u(t)\|_H^2 = \limsup_{t \rightarrow \infty} \int_H \|u\|_H^2 d\mathcal{T}(t) \mu_0 \leq \frac{c_{24} + \operatorname{tr}_{L_2} Q}{c_{25}}$$

if the initial distribution μ_0 of the random initial condition $u_0(\omega)$ is contained in M^2 . Here $c_{25} > 0$ also depends on physical data. In particular, we have asymptotic mean-square estimate for the atmospheric temperature evolution under oceanic feedback

$$\limsup_{t \rightarrow \infty} \mathbb{E}\|\Theta\|_H^2 \leq \frac{c_{24} + \operatorname{tr}_{L_2} Q}{c_{25}}. \tag{24}$$

Thus the atmospheric temperature $\Theta(y, t)$, as modeled by the coupled atmosphere-ocean system (1), is bounded asymptotically in mean-square norm in terms of physical quantities such as the freshwater flux $F(y)$, the trace of the covariance operator of the external noise, the earth’s longwave radiative cooling coefficient a , and the empirical functions $S_a(y)$ and $S_o(y)$ representing the latitudinal dependence of the shortwave solar radiation, as well as the Prandtl number Pr and the Rayleigh number Ra for oceanic fluids.

By the estimates of Theorem 5.12, we are able to use the well-known Krylov–Bogoliubov procedure to conclude the existence of invariant measures of the Markov semigroup.

Corollary 5.14. *The semigroup of Markov operators $\{\mathcal{T}(t)\}_{t \geq 0}$ possesses an invariant distribution μ_I in M^2 :*

$$\mathcal{T}(t)\mu_I = \mu_I \text{ for } t \geq 0.$$

In fact, the limit points of

$$\left\{ \frac{1}{t} \int_0^t \mathcal{T}(\tau)\mu_0 d\tau \right\}_{t \geq 0}$$

for $t \rightarrow \infty$ are invariant distributions. The existence of such limit points follows from the estimate in Theorem 5.12.

In some situations, the invariant measure may be unique. For example, the unique random fixed point in Corollary 5.11 is defined by a random variable $u^*(\omega) = v^*(\omega) + z(\omega)$. This random variable corresponds to a unique invariant measure of the Markov semigroup. More specifically, this unique invariant measure is the expectation of the Dirac measure with the random variable as the random mass point

$$\mu_i = \mathbb{E}\delta_{u^*(\omega)}.$$

As the uniqueness of invariant measure implies ergodicity [21], we conclude that the coupled atmosphere-ocean model (1) is ergodic under the suitable conditions in Corollary 5.11 for physical data and random noise. We reformulate Corollary 5.11 as the following ergodicity principle.

Theorem 5.15. (Ergodicity) *Assume that the physical data $|a|$, $\|S_a\|_{L_2}$, $\|S_o\|_{L_2}$, $\|F\|_{L_2}$ and the trace of the covariance for the noise $\text{tr}_H Q$ are sufficiently small, and that the viscosity ν is sufficiently large. Then the coupled atmosphere-ocean system (1) is ergodic, namely, for any observable of the coupled atmosphere-ocean flows, its time average approximates the statistical ensemble average, as long as the time interval is sufficiently long.*

6. Summary

We have investigated the dynamical behavior of a coupled atmosphere-ocean model. First, we have shown that the asymptotic dynamics of the coupled atmosphere-ocean model is described by a random climatic attractor (Corollary 5.3). Second, we have estimated the atmospheric temperature evolution under oceanic feedback, in terms of the freshwater flux, heat flux and the external fluctuation at the air-sea interface, as well as the earth's longwave radiation coefficient and the shortwave solar radiation profile (Corollary 5.13). Third, we have demonstrated that this system has finite degree of freedom by presenting a finite set of determining functionals in probability (Corollary 5.7). Finally, we have proved that the coupled atmosphere-ocean model is ergodic under suitable conditions for physical parameters and randomness, and thus for any observable of the coupled atmosphere-ocean flows, its time average approximates the statistical ensemble average, as long as the time interval is sufficiently long (Theorem 5.15).

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