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## Ergodic dynamics of the stochastic Swift–Hohenberg system<sup>☆</sup>

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### Abstract

The Swift–Hohenberg fluid convection system with both local and nonlocal nonlinearities under the influence of white noise is studied. The objective is to understand the difference in the dynamical behavior in both local and nonlocal cases. It is proved that when sufficiently many of its Fourier modes are forced, the system has a unique invariant measure, or equivalently, the dynamics is ergodic. Moreover, it is found that the number of modes to be stochastically excited for ensuring the ergodicity in the local Swift–Hohenberg system depends *only* on the Rayleigh number (i.e., it does not even depend on the random term itself), while this number for the nonlocal Swift–Hohenberg system relies additionally on the bound of the kernel in the nonlocal interaction (integral) term, and on the random term itself.

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## 1. Introduction

looseness=1Density gradient-driven fluid convection arises in geophysical fluid flows in the atmosphere, oceans and the earth's mantle. The Rayleigh–Benard convection is a prototypical model for fluid convection, aiming at predicting spatio-temporal convection patterns. The mathematical model for the Rayleigh–Benard convection involves the Navier–Stokes equations coupled with the transport equation for temperature. When the Rayleigh number is near the onset of the convection, the Rayleigh–Benard convection model may be approximately reduced to an amplitude or order parameter equation, as derived by Swift and Hohenberg [34].

In the literature, most works (e.g., [17,14,28]) on the Swift–Hohenberg model deal with the following evolution equation for order parameter  $u(x, t)$ , which is localized version of the model originally derived by Swift and Hohenberg [34],

$$u_t = \varrho u - (1 + \partial_{xx})^2 u - u^3,$$

where  $\varrho$  measures the difference of the Rayleigh number from its critical onset value and the cubic term  $u^3$  is an yet “approximation” of a nonlocal integral term in the original Swift–Hohenberg model [34].

Roberts [32,33] re-examined the rationale for using the Swift–Hohenberg model as a reliable simplified model of the spatial pattern evolution in fluid convection. He argued that, although the localization approximation used in (1) makes some sense, the approximation is deficient in describing some basic features of such systems, and devised via symmetry argument the following modified Swift–Hohenberg equation with nonlocal interactions:

$$u_t = \varrho u - (1 + \partial_{xx})^2 u - uG * u^2,$$

where  $G * u^2$  is a spatial convolution integral and  $G(\cdot)$  is a given radially symmetric positive (or nonnegative) function. We call this function  $G$  the kernel for the nonlocal term. In fact, nonlocal integral terms often appear naturally in amplitude equation models for nonequilibrium systems; see, e.g., [25,15,8,23].

Our goal in this paper is to examine the above local and nonlocal Swift–Hohenberg models, by investigating the dynamical difference of both models under random impact as well as under nonlocal interactions.

Fluid systems are often subject to random environmental influences. On the one hand, there is a growing recognition of a role for the inclusion of stochastic effects in the modeling of complex systems. Randomness can have delicate impact on the overall evolution of such systems, for example, noise-induced phase transition or stochastic bifurcation [4], stochastic resonance [19], and noise-induced pattern formation [13,2]. Taking stochastic effects into account is of central importance for the development of mathematical models of such complex phenomena in engineering and science. Macroscopic models in the form of partial differential equations for these systems contain such randomness as stochastic forcing, uncertain parameters, random sources or inputs, and random boundary conditions. Stochastic partial differential equations (SPDEs) are appropriate models for randomly influenced spatially extended systems. On the other hand, the inclusion of such effects has led to interesting new mathematical problems at the interface of probability and partial differential equations.

In this paper, we consider the Swift–Hohenberg model, taking stochastic forcing as well as nonlocal interactions, into account.

First, we consider the local stochastic Swift–Hohenberg system (LSSH)

$$u_t = \varrho u - (1 + \partial_{xx})^2 u - u^3 + F(x, t), \quad u(0) = u_0, \tag{1}$$

under the periodic boundary condition with period  $2\pi$ , in which the stochastic force is taken to be of the form

$$F(x, t) = \partial_t W(x, t) = \partial_t \sum_{i=1}^{N'} b_i e_i(x) w_i(t), \quad N' \leq \infty, \tag{2}$$

where  $\{e_i\}$  is an orthonormal basis of the Hilbert space  $H = L^2_{\text{per}}(0, 2\pi)$  formed by the eigenvectors of the operator  $A = -(1 + \partial_{xx})^2$ ,  $\{w_i(t), t \geq 0\}$  are independent standard Wiener processes, and the real coefficients  $b_i \geq 0$  satisfies

$$b_i \neq 0, \quad 1 \leq i \leq N \leq N'$$

for some sufficiently large  $N$ .

Then, we study the nonlocal stochastic Swift–Hohenberg system (NLSSH)

$$u_t = \varrho u - (1 + \partial_{xx})^2 u - uG * u^2 + \tilde{F}(x, t), \quad u(0) = u_0, \tag{3}$$

with a positive kernel (i.e.,  $G > 0$ ) and a special nonnegative kernel (i.e.,  $G \geq 0$ ), respectively, under the periodic boundary condition, in which

$$G * u^2 = \int_0^{2\pi} G(x - \xi) u^2(\xi, t) d\xi$$

and the stochastic force is taken to be of the form

$$\tilde{F}(x, t) = \partial_t \tilde{W}(x, t) = \partial_t \sum_{i=1}^{N'} \tilde{b}_i e_i(x) w_i(t), \quad N' \leq \infty. \tag{4}$$

The diagnostic tool we choose to compare the dynamical differences between the local and nonlocal stochastic Swift–Hohenberg systems is *ergodicity*. Namely, we are interested in the ergodicity of the Markov solution process in  $H$  generated by the stochastic Swift–Hohenberg equation (1) which forms a random dynamical system (RDS). More precisely, we first prove that if sufficiently many of its Fourier modes are forced, the LSSH system has a unique invariant measure, or equivalently, the dynamics is ergodic in the phase space, and that all solutions converge exponentially fast in distribution to this unique measure (Theorem 5.2 and Corollary 5.3). Our results show that under certain conditions the RDS generated by (1) has a global exponentially attracting fixed point in the sense of distribution. And this fixed point is the unique invariant measure which is supported on the random attractor. Namely, the LSSH system (1) is ergodic. To prove our results, we decompose the LSSH system (1) into a low mode part with a finite dimension and a high mode part with an infinite dimension. In the low mode part, we use the maximal coupling approach, which is used

in [21], to prove that the low mode part can be coupled. The high mode part can be enslaved, and then the ergodic result is obtained; see also [3,16,27].

We verify that the dynamics of the nonlocal stochastic Swift–Hohenberg equation (3) is also ergodic, provided sufficiently many of its Fourier modes are forced (Theorems 6.4 and 6.5). However, our interest here is to investigate the difference in the conditions ensuring the ergodicity of the LSSH and NLSSH systems. We find out that (see discussions in Section 7):

(i) For the *local* Swift–Hohenberg system, the estimated number of Fourier modes to be randomly excited for ensuring ergodicity depends only on the parameter  $\varrho$ , which measures the difference of the Rayleigh number from its critical onset value. Note that this number does not depend on the random forcing term;

(ii) For the *nonlocal* Swift–Hohenberg system with positive kernel  $G$  in the nonlocal nonlinearity, the estimated number of Fourier modes to be randomly excited for ensuring ergodicity depends additionally on the upper and lower bounds of the positive kernel, and on the random term itself.

Recently, there have been a number of papers on ergodicity of stochastically forced partial differential equations (SPDEs) and, in particular, 2D Navier–Stokes equations. The stochastic force of the SPDEs may be in the form of (2) above or in the following form (so-called kick-force):

$$F(x, t) = \sum_{k=-\infty}^{\infty} f_k \delta(t - Tk), \quad f_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \tag{5}$$

where  $b_j \geq 0$  are some constants such that  $\sum b_j^2 < \infty$  and  $\{\xi_{jk}\}$  are independent random variables with  $k$ -independent distributions, or the white (in time) force of form (2). A coupling approach and the so-called Kantorovich functionals (e.g., [20,22,27]) have been developed to show exponential convergence to the unique invariant measure for stochastic Navier–Stokes equation. Similar results were obtained in [30]. For the white noise forced PDE, [11] and [26] obtained the ergodicity of the stochastic forcing Navier–Stokes equation for the case when the random force is singular in  $x$ . For the 2D Navier–Stokes equation with periodic boundary condition and random forcing, [9] proved uniqueness of the stationary measure under the condition that all “deterministic modes” are forced, by studying the Gibbsian dynamics of the low modes. The idea in [9] has been used in other papers to get explicit results. Moreover, [16] obtained the exponential mixing property of the stochastic Ginzburg–Landau equation by the so-called *binding construction*. We also mention that [21] developed the idea in [20,22] to get the rates of the measure converge to the unique invariant measure for 2D Navier–Stokes equation with a white noise to all Fourier modes.

In the present paper, the linear part  $-(1 + \partial_{xx})^2 u$  in the Swift–Hohenberg models is not dissipative, i.e., it has positive eigenvalues. Similar situation has also been considered in [16], for example. In this case, the coupling approach can still be used to investigate ergodicity. In fact, we can directly study the maximal coupling solutions of the stochastic Swift–Hohenberg system, without constructing the Kantorovich functional; see [16,27]. Then the key fact that the orbit in set  $S(m, k)$  comes from  $R(m) \cup S(m, m)$  helps us to prove the main result without using the exponential decay of a Kantorovich-type functional, for details see Sections 4 and 5. For simplicity of presentation, in this paper we only work with

the periodic boundary condition and only consider the case when only finite modes are forced, i.e.,  $N' < \infty$ . However, we emphasize that our argument applies in principle to the case when the other more physical boundary conditions are posed and when all the modes are subject to random excitation. Moreover, our techniques also applies to more realistic 2D Swift–Hohenberg model, i.e.,  $(1 + \partial_{xx})^2$  is replaced by  $(1 + \partial_{xx} + \partial_{yy})^2$ .

We remark that a systematic treatment for random dynamical systems (including ergodicity) is emerging; see, e.g., [1,7,5,35,36,37].

The remainder of this paper is organized as follows. In Section 2, we decompose the local system LSSH into the low mode part and the high mode part. Section 3 is devoted to some energy estimates of the solution. The coupling construction and a coupling result are given in Section 4. Then we give the ergodic result in Section 5. In Section 6, we study the ergodicity of nonlocal system NLSSH with positive or nonnegative kernels of nonlocal nonlinearity, respectively, by the techniques developed in the previous sections. Finally, we discuss the difference between LSSH system and NLSSH system by analysing the minimal conditions ensuring the ergodicity in Section 7.

## 2. Decomposition of the local system

We consider the following system on the real line:

$$u_t - qu + (1 + \partial_{xx})^2 u + u^3 = F(x, t), \tag{6}$$

$$u(0) = u_0, \tag{7}$$

$$u(t, x) = u(t, x + 2\pi). \tag{8}$$

Let  $A = -(1 + \partial_{xx})^2$ . It is well known that  $A$  generates a compact analytic semigroup  $e^{tA}$  in  $C_{\text{per}}^0$ . Since

$$B_0 = \sum_{i=1}^{N'} b_i^2 < \infty, \tag{9}$$

the stochastic convolution  $\int_0^t e^{A(t-\tau)} dW(x, t)$  is continuous for  $(x, t)$ ; see [6].

For convenience, we assume  $N' < \infty$ . Let  $\langle \cdot, \cdot \rangle$  be the scalar product of  $H = L_{\text{per}}^2(0, 2\pi)$ ,  $\|\cdot\|$  the norm, and  $V = H^1 \cap H$ . Let

$$H_l = \text{span}\{e_i, 0 < i \leq N\}, \quad H_h = \text{span}\{e_i, i > N\},$$

which are two subspaces of  $H$ . We will call  $H_l$  the low mode space and  $H_h$  the high mode space. Clearly,  $H = H_l \oplus H_h$ . Let  $P_l$  and  $P_h$  be the orthogonal projections onto  $H_l$  and  $H_h$ , respectively. Denoting  $u(t) = (l(t), h(t))$ , then (6)–(8) is equivalent to the following equations on  $H$ :

$$\dot{l} = P_l A l + q l - P_l (l + h)^3 + P_l F(x, t), \quad l(0) = l_0; \tag{10}$$

$$\dot{h} = P_h A h + q h - P_h (l + h)^3, \quad h(0) = h_0, \tag{11}$$

where  $l_0 = P_l u(0)$  and  $h_0 = P_h u(0)$ . For any given  $T > 0$ , since the noise is additive we can treat the equation pathwise. Then the deterministic theory implies that (6)–(8) admits a

mild solution with transition probability  $P(t, x, \Gamma)$  for  $\Gamma \in \mathcal{B}(H)$ , the Borel  $\sigma$ -algebra in  $H$ . Here, we treat (6)–(8) in another way that for  $l_0 \in H_l, h_0 \in H_h$ , (10)–(11) has a unique solution  $(l, h) \in X = C([0, T], H)$ , where  $l \in C(0, T; H_l)$  and  $h \in C(0, T; H_h)$ . Notice that, for given  $l(t)$  we can solve (10) with  $h_0 \in H_h$ . This solution is written as  $\Phi(l, h_0)$ . Much more such decomposition is discussed in [10].

Let  $l_1 = l, h_1 = h$  for  $t \in [0, T]$ ,  $l_k$  and  $h_k$  be the solutions on  $[0, kT]$ ,  $l_{m,k}$  and  $h_{m,k}$  be the solutions on  $[mT, kT]$  for  $0 \leq m \leq k$ . Similar for  $u_m$  and  $u_{m,k}$ .

For (10) and (11), we have the following proposition, which displays the main idea of the paper.

**Proposition 2.1.** *For any  $u_0^1, u_0^2 \in H$ , if  $l_{m,k}^1 = l_{m,k}^2$  then there exists a constant  $\lambda > 0$  such that*

$$|u^1 - u^2| \leq e^{-\lambda(t-mT)} |u^1(mT) - u^2(mT)|, \quad t \in [mT, kT].$$

**Proof.** Since  $l_{m,k}^1 = l_{m,k}^2$ , we only consider the high mode. Let  $\rho = h^1 - h^2$ . Then  $\rho$  satisfies

$$\dot{\rho} = A\rho + \varrho\rho - P_h((u^1)^3 - (u^2)^3).$$

Taking the scalar product of the above equation with  $\rho$  in  $H_h$  and notice that

$$\langle P_h((u^1)^3 - (u^2)^3), \rho \rangle \geq 0,$$

we have

$$\partial_t |\rho|^2 \leq (\alpha_N + \varrho) |\rho|^2,$$

where  $\alpha_N$  is the  $N$ th eigenvalue of  $A$ . If we choose  $N$  so large that  $\alpha_N + \varrho < 0$ , then

$$|\rho|^2 \leq e^{-\lambda(t-mT)} |\rho(mT)|, \quad t \in [mT, kT],$$

for some constant  $\lambda$ . This completes the proof.  $\square$

**Remark 2.2.** This proposition implies that, if we want to get the distributions of the LSSH system converged to the unique invariant measure, it is enough to prove the following: for any two solutions  $u^1$  and  $u^2$ , the probability of the event that those low modes coincide is exponentially tends to 1, as  $t \rightarrow \infty$ . But, in fact, we do not need that all solutions satisfy the above proposition. By the coupling result in Lemma 4.1 below, we just need that the solutions that do not grow fast satisfy the proposition. This will also be used for the NLSSH system.

### 3. Some estimates

In this section, we derive some estimates of the solutions for the LSSH system, which will be used in the following. We will work on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  generated by  $\{W(t)\}$ . We associate  $\Omega$  with the canonical space generated by all  $dw_i(t)$ . Denote  $\mathcal{F}$  as

the associated  $\sigma$ -algebra generated by  $\{W(t)\}$  with  $\mathbb{P}$  the probability measure. Expectations with respect to  $\mathbb{P}$  will be denoted by  $\mathbf{E}$ .

The following lemma describes the growth rate of  $|u(t)|^2$ .

**Lemma 3.1.** *There is a positive constant  $C_1$ , such that for any  $r > 0$ ,*

$$\mathbb{P} \left\{ |u(t)|^2 + \int_0^t |u(s)|^2 ds \leq |u(0)|^2 + C_1 t + r, \text{ for } t \geq 0 \right\} \geq 1 - e^{-r}.$$

**Proof.** Applying Ito’s formula to  $|u(t)|^2$ , one yields

$$\frac{1}{2} \frac{d}{dt} |u(t)|^2 = \langle Au, u \rangle + \langle \varrho u, u \rangle - \langle u^3, u \rangle + \frac{1}{2} B_0 + \langle F(x, t), u \rangle.$$

Note that

$$\begin{aligned} \langle Au + \varrho u, u \rangle &= \langle -(1 - \varrho)u - 2\partial_{xx}u - \partial_{xx}^2 u, u \rangle \\ &\leq -(1 - \varrho)|u|^2 + |\partial_{xx}u|^2 + |u|^2 - |\partial_{xx}u|^2 \\ &= \varrho|u|^2 \end{aligned}$$

and

$$|u| \leq (2\pi)^{1/4} |u|_{L^4}.$$

Then we have

$$\begin{aligned} |u|^2 + \int_0^t |u(s)|^2 ds &\leq |u(0)|^2 + \int_0^t \left[ 2 \left( \varrho + \frac{1}{2} \right) |u|^2 - \frac{1}{2\pi} |u|^4 + B_0 \right] ds + 2 \int_0^t \langle F, u \rangle \\ &= |u(0)|^2 + \int_0^t \left[ 2 \left( \varrho + b_{\max}^2 + \frac{1}{2} \right) |u|^2 - \frac{1}{2\pi} |u|^4 + B_0 \right] ds + M_t \\ &\quad - \langle M \rangle_t / 2 - \left( \int_0^t 2b_{\max}^2 |u|^2 ds - \langle M \rangle_t / 2 \right), \end{aligned}$$

where  $b_{\max} = \max_i b_i$ ,  $M_t = 2 \int_0^t \langle F, u \rangle$  and  $\langle M \rangle_t$  is the quadratic variation.

Note that

$$\langle M \rangle_t / 2 = 2 \sum_i b_i^2 \int_0^t u_i^2 ds \leq 2b_{\max}^2 \int_0^t |u|^2 ds.$$

That is,

$$|u(t)|^2 + \int_0^t |u(s)|^2 ds \leq |u(0)|^2 + C_1 t + M_t - \langle M \rangle_t / 2,$$

where  $C_1 = \max_x \{ 2(\varrho + b_{\max}^2 + \frac{1}{2})x^2 - \frac{1}{2\pi}x^4 + B_0 \}$ .

Then the classical supermartingale inequality implies

$$\begin{aligned} & \mathbb{P} \left\{ |u(t)|^2 + \int_0^t |u(s)|^2 ds \leq |u(0)|^2 + C_1 t + r \right\} \\ & \geq \mathbb{P} \{ M_t - \langle M \rangle_t / 2 \leq r \} \\ & \geq \mathbb{P} \{ \exp(M_t - \langle M \rangle_t / 2) \leq e^r \} \\ & \geq 1 - e^{-r}. \end{aligned}$$

This completes the proof.  $\square$

We also need the following estimation of the mean value of the solution.

**Lemma 3.2.** *For any  $t \geq 0$ , we have*

$$\mathbf{E}|u(t)|^2 \leq e^{-\alpha t} \mathbf{E}|u_0|^2 + R,$$

where  $\alpha$  and  $R$  are some positive constants.

**Proof.** Applying Itô formula to  $|u|^2$  and taking the mean value, we find

$$\begin{aligned} \mathbf{E}|u|^2 + \alpha \int_0^t \mathbf{E}|u|^2 ds & \leq \mathbf{E}|u_0|^2 + \int_0^t [(2\varrho - \alpha)\mathbf{E}|u|^2 - \frac{1}{2\pi} \mathbf{E}|u|^4 + B_0] ds \\ & \leq \mathbf{E}|u_0|^2 + R\alpha t, \end{aligned}$$

where  $\alpha$  is some positive number and  $R\alpha = \max_x \{(2\varrho - \alpha)x^2 - \frac{1}{2\pi}x^4 + B_0\}$ . Then Gronwall inequality yields the results.  $\square$

#### 4. Coupling approach for ergodicity

We start this section with some notations and terminology. Let  $H$  be a separable Hilbert space with  $\sigma$ -algebra  $\mathcal{B}(H)$  and let  $\mathcal{M}(H)$  be the space of sign measures with bounded variation. We denote by  $\mathcal{P}(H)$  the set of probability measures  $\mu \in \mathcal{M}(H)$ . For  $\mu \in \mathcal{P}(H)$  we define the variation norm as

$$\|\mu\|_{\text{var}} = \sup_{A \in \mathcal{B}(H)} |\mu(A)|.$$

If  $\mu_1, \mu_2 \in \mathcal{P}(H)$  are absolutely continuous with respect to a fixed Borel measure  $m$ , then we have

$$\|\mu_1 - \mu_2\|_{\text{var}} = \frac{1}{2} \int_H |\rho_1(u) - \rho_2(u)| dm(u),$$

where  $\rho_i, i = 1, 2$ , is the density of  $\mu_i$  with respect to  $m$ . For more see [20].

The coupling is a well-known effective tool for studying finite-dimensional Markov chains [24]. To our knowledge, [29] is the first paper using a coupling approach for invariant measures of the stochastic heat equation. The main idea of coupling is to construct two

random variables  $\xi_1$  and  $\xi_2$ , for two measures  $\mu_1$  and  $\mu_2$  that we are concerned with, and study the two measures through constructing the two random variables. In this section, we will use the maximal coupling, namely, we construct  $\xi_1$  and  $\xi_2$  such that

$$\|\mu_1 - \mu_2\|_{\text{var}} = \mathbb{P}\{\xi_1 \neq \xi_2\}.$$

About the maximal coupling, we refer to [20,24].

In this section, we always use the maximal coupling  $(l_1^1, l_1^2)$  for the distributions of  $(P_l u^1, P_l u^2)$  on  $C(0, T; H_l)$ . By the discussion in Section 2, we can define the coupling solution  $u^i(t)$  as

$$u^i(t) = (l_k^i(t), \Phi(l_k^i, h^i((k-1)T))), \quad t \in [(k-1)T, kT], \quad k \geq 1, \quad i = 1, 2$$

for (5)–(7) on the product space  $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k) \times X^k$ , where

$$\Omega^k = \Omega \times \dots \times \Omega, \quad \mathcal{F}^k = \mathcal{F} \times \dots \times \mathcal{F}, \quad \mathbb{P}^k = \mathbb{P} \times \dots \times \mathbb{P}.$$

and  $X^k = X \times \dots \times X$ . From Theorem 4.2 in [20], it follows that  $u^i$  is measurable coupling solutions of (6)–(8). Notice that, the coupling solutions are not the solutions of (6)–(8), but they have the same distributions with the solutions of (6)–(8), since we just change  $F(x, t)$  as another noise with the same distribution. Then, we will call the coupling solutions the weak solutions as in [21]. Obviously, the results in Section 2 still hold true if  $u(t)$  is a weak solution of (6)–(8).

Now, for those weak solutions we define some function spaces which play a role in our approach. This arises from Remark 2.2. These spaces are also introduced in [21]. For any given real number  $T > 0$  and integer number  $k \geq 1$ , we define  $S_0(m, k)$  as the set of functions  $(u^1(t), u^2(t)), t \in [mT, kT], 0 \leq m \leq k$ , such that

$$u^i \in L^2(0, kT; V) \cap C(0, kT; H), \tag{12}$$

$$P_l u^1(t) = P_l u^2(t), \quad |u^1(mT)| \vee |u^2(mT)| \leq D, \tag{13}$$

$$\mathcal{E}^i(t, mT) \leq r + (C_1 + 1)(t - mT), \quad mT \leq t \leq kT, \quad i = 1, 2, \tag{14}$$

where  $a \vee b = \max(a, b)$  for  $a, b \in \mathbb{R}$  and  $\mathcal{E}^i(t, mT) = |u^i(t)|^2 + \int_{mT}^t |u^i(s)|^2 ds$ . The set  $S_0(m, k)$  is called the coupled set. Then the set

$$R(k) = (C(0, kT; H) \cap L^2(0, kT; V)) \bigcap_{m=0}^k S_0(m, k)$$

should be the exponentially small set by Proposition 2.1.

Let

$$S(m, k) = S_0(m, k) \setminus S(m-1, k), \quad 0 \leq m \leq k,$$

where  $S(-1, k)$  is the null set. In the following, we always assume  $|u_0^1|^2$  and  $|u_0^2|^2$  have the finite mean value. We observe that the orbit in  $S(m, k-1)$  enters either  $S(m, k)$  or  $R(k) \cup S(k, k)$  for  $0 \leq m \leq k-1$ , and the orbit in  $S(m, k)$  comes from the set  $R(m) \cup S(m, m)$ ; see Fig. 1.

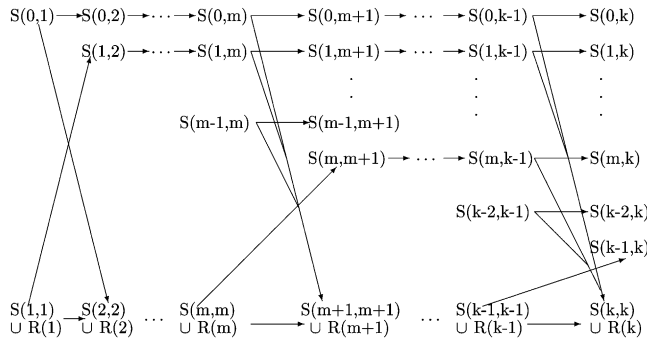


Fig. 1. Orbit transition.

Then, what needs to be proved is that the event that  $u^1$  and  $u^2$  are in  $R(k) \cup S(k, k)$  has exponentially small probability. Clearly, this demands that the events that the orbits in  $S(m, k - 1)$  enter  $R(k) \cup S(k, k)$  or do not enter  $S(m, k)$  have exponentially small probability.

**Lemma 4.1.** Fix  $T > 0$  large enough. Let  $\lambda_1$  and  $\lambda_2$  be the distributions of  $l_{m,k}^1$  and  $l_{m,k}^2$ , respectively, for  $(u^1, u^2) \in S(m, k - 1)$ ,  $0 \leq m < k - 1$ . If (1) holds and  $N$  is large enough, then

$$\|\lambda_1 - \lambda_2\|_{\text{var}} \leq e^{-\gamma(k-m)T}$$

for some positive constant  $\gamma$ .

**Proof.** Let  $l^1$  and  $l^2$  be the maximal coupling for  $\lambda_1$  and  $\lambda_2$ , respectively. There are two cases to be distinguished for  $t \in [0, T]$ .

Case 1:  $\mathcal{E}^i((k - 1)T + t, mT) < r + (C_1 + 1)((k - 1)T + t - mT)$ ,  $i = 1, 2$ .

Let

$$l_0^i = P_l u^i(mT), \quad h_0^i = P_h u^i(mT), \quad i = 1, 2.$$

Then  $l_0^1 = l_0^2 = l$ . Consider the LSSH system in low mode space

$$\frac{dl^i(t)}{dt} = [Al^i + ql^i - P_l(l^i + h^i)^3] + P_l F(x, t), \quad i = 1, 2.$$

Let  $B(l, h_0^1, h_0^2) = P_l(l + h^1)^3 - P_l(l + h^2)^3$ . Then

$$\begin{aligned} |B(l, h_0^1, h_0^2)|^2 &\leq \sup_{y \in L^2_\gamma \text{ and } |y|=1} |\langle P_l(l + h^1)^3 - P_l(l + h^2)^3, y \rangle|^2 \\ &\leq |(l + h^1)^3 - (l + h^2)^3|^2 \\ &\leq |h^1 - h^2|^2 |(u^1)^2 + (u^1)(u^2) + (u^2)^2|^2 \end{aligned}$$

Proposition 2.1 tells us that

$$|h^1 - h^2|^2 \leq 4D^2 e^{-\lambda((k-1)T+t-mT)}.$$

Then, there exists some positive constant  $\lambda'$ , such that

$$|B(l, h_0^1, h_0^2)|^2 \leq D^2 e^{-\lambda'(k-1-m)T},$$

since  $|u^i(t)|^2$  at most increases polynomially. Thus

$$\int_0^T |B(l, h_0^1, h_0^2)|^2 dt \leq K_0 e^{-\lambda'(k-m-1)T} \tag{15}$$

for some constant  $K_0 > 0$ . Now we apply the Girsanov’s formula to  $\lambda_i$ . Let

$$\beta(t, \omega) = \mathbf{b}^{-1} B(l, h_0^1, h_0^2),$$

where  $\mathbf{b}$  is the  $N \times N$  diagonal matrix with diagonal elements  $b_j, j = 1, \dots, N$ . Then, Girsanov’s formula yields

$$\lambda_1(dl) = e^{G(l)} \lambda_2(dl), \tag{16}$$

where

$$G(l) = - \int_0^T (\beta, \mathbf{b}^{-1} dw_t) - \frac{1}{2} \int_0^T |\beta|^2 dt.$$

For (13)

$$\begin{aligned} \mathbf{E} \exp(2G(l)) &= \mathbf{E} \exp \left( -2 \int_0^T (\beta, \mathbf{b}^{-1} dw_t) - \int_0^T |\beta|^2 dt \right) \\ &\leq \left( \mathbf{E} \exp \left( -4 \int_0^T (\beta, \mathbf{b}^{-1} dw_t) - 8 \int_0^T |\beta|^2 dt \right) \right)^{1/2} \\ &\quad \times \left( \mathbf{E} \exp \left( 6 \int_0^T |\beta|^2 dt \right) \right)^{1/2} \\ &\leq e^{3K_N D^2 e^{-\lambda'(k-m-1)T}} \end{aligned}$$

with some constant  $K_N$ . Hence,

$$\begin{aligned} \|\lambda_1 - \lambda_2\|_{\text{var}} &= \frac{1}{2} \int_{C(0,T;H)} \left| 1 - \frac{d\lambda_2}{d\lambda_1} \right| d\lambda_1 \\ &\leq \frac{1}{2} \left( \int_{C(0,T;H)} \left| 1 - \frac{d\lambda_2}{d\lambda_1} \right|^2 d\lambda_1 \right)^{1/2} \\ &\leq \frac{1}{2} (e^{3K_N D^2 e^{-\lambda'(k-m-1)T}} - 1)^{1/2} \\ &\leq e^{-\gamma_0(k-m)T} \end{aligned}$$

for some positive constant  $\gamma_0$ , since  $T > 0$  is large enough.

Case 2:  $\mathcal{E}^i((k - 1)T + t, mT) \geq r + (C_1 + 1)((k - 1)T + t - mT)$  for some  $t, i = 1, 2$ . This is the direct result of Lemma 3.1. In fact, the inequality

$$\mathcal{E}^i((k - 1)T + t, mT) \geq r + (C_1 + 1)((k - 1)T + t - mT)$$

implies that

$$\begin{aligned} \mathcal{E}^i((k - 1)T + t, mT) &\geq |u(mT)|^2 + C_1((k - 1)T + t - mT) + (r - |u(mT)|^2) \\ &\quad + (k - m - 1)T. \end{aligned}$$

Taking  $\gamma = \min\{1, \gamma_0\}$  completes the proof.  $\square$

**Remark 4.2.** Lemma 4.1 implies that, the probability of the event that the orbits in  $S(m, k - 1), 0 \leq m < k - 1$  enter  $R(k) \cup S(k, k)$  is small.

To our aim we also need the following lemma.

**Lemma 4.3.** *Let (1) hold. If  $(u^1, u^2) \in R(k - 1)$  satisfies  $|u^1((k - 1)T)| \vee |u^2((k - 1)T)| \leq D$ , then*

$$\mathbb{P}\{(u^1, u^2) \in S(k, k)\} \geq d^* > 0$$

for some constant  $d^* > 0$  depending only on  $D$ .

**Proof.** From Lemma 3.1, we have

$$\mathbb{P}\{\mathcal{E}^i(t, (k - 1)T) \leq r + (C_1 + 1)(t - (k - 1)T)\} \geq c^* \tag{17}$$

for some positive constant  $c^*$  which depends only on  $D$ . For the purpose of this lemma, we compare  $l^i(t)$  with the following standard diffusion process:

$$\dot{\tilde{l}}(t) = A\tilde{l} + (q - v)\tilde{l} + P_l F(x, t). \tag{18}$$

Here,  $v$  is large enough such that (18) is linearly dissipative. Let  $\tilde{l}^i$  be the solutions of (18) with the initial value  $l^i((k - 1)T)$ . Let  $\lambda_i(t)$  and  $\tilde{\lambda}_i(t)$  be the distribution of  $l^i_{k-1,k}$  and  $\tilde{l}^i_{k-1,k}$  restricting on  $C((k - 1)T, t; H_l)$ , respectively, for  $i = 1, 2$ . Since (18) is linearly dissipative, we have

$$\int \left[ \frac{d\tilde{\lambda}_1}{d\tilde{\lambda}_2} \right]^2 d\tilde{\lambda}_1 \leq c_1^*$$

for some constant  $c_1^* > 0$  which depends only on the upper bound  $D$ .

By the similar method in the proof of Lemma 4.1, we have

$$\int \left[ \frac{d\lambda_i}{d\tilde{\lambda}_i} \right]^2 d\tilde{\lambda}_i \leq c_2^* \quad i = 1, 2$$

for some positive constant  $c_2^*$  which depends only on  $D$ . Since at this time  $|B^i| = |v\tilde{v}^i + P_l(l^i + h^i)^3|$  is bounded as we consider the event of (17) (see [10]). By Hölder inequality, we can derive

$$\int \left[ \frac{d\lambda_1}{d\lambda_2} \right]^2 d\lambda_1 \leq c_3^*$$

for some positive constant  $c_3^*$  which depends only on  $D$ . Then by the Lemma C.1 of [27], we have

$$\int \left| 1 \wedge \frac{d\lambda_1}{d\lambda_2} \right| d\lambda_2 \geq c_4^*$$

for some constant  $c_4^* > 0$  depending only on  $D$ . Notice that  $\|\lambda_1 - \lambda_2\|_{\text{var}} = 1 - \|\lambda_1 \wedge \lambda_2\|_{\text{var}}$ , we have

$$\|\lambda_1 - \lambda_2\|_{\text{var}} \leq d^* = 1 - c_4^*.$$

Then  $\mathbb{P}\{(u^1, u^2) \in S(k, k)\} \geq 1 - d^*$ . This completes the proof.  $\square$

Then, we have:

**Proposition 4.4.** *Let (1) hold. For  $(u^1, u^2) \in R(k-1) \cup S(k-1, k-1)$ , there is a constant  $0 < \delta < 1$ , such that*

$$\mathbb{P}\{(u^1, u^2) \in R(k)\} \leq \delta \mathbb{P}\{(u^1, u^2) \in R(k-1) \cup S(k-1, k-1)\},$$

**Proof.** This is the direct result of Lemma 4.3. We omit the proof.  $\square$

By Lemma 4.1 and Proposition 4.4, we have the following important proposition, which implies the existence of a unique invariant measure.

**Proposition 4.5.** *Suppose (1) holds. There is a constant  $0 < \kappa < 1$  such that for any initial value  $u_0^1, u_0^2 \in H$ ,*

$$\mathbb{P}\{(u^1, u^2) \in R(k) \cup S(k, k)\} \leq K \kappa^k,$$

where  $K$  is a constant depending only on the initial value.

**Proof.** We will consider  $R(k)$  and  $S(k, k)$ , respectively. First, we have

$$\begin{aligned} & \mathbb{P}\{(u^1, u^2) \in R(k)\} \\ & \leq e^{-\gamma k T} \sum_{m=0}^{k-2} e^{\gamma m T} \mathbb{P}\{(u^1, u^2) \in S(m, k-1)\} \\ & \quad + \delta \mathbb{P}\{(u^1, u^2) \in S(k-1, k-1) \cup R(k-1)\} \\ & \leq (2\delta e^{\gamma T} + 1)e^{-\gamma k T} \sum_{m=0}^{k-3} e^{\gamma m T} \mathbb{P}\{(u^1, u^2) \in R(m)\} \\ & \quad + (2\delta e^{\gamma T} + 1)e^{-\gamma k T} \sum_{m=0}^{k-3} e^{\gamma m T} \mathbb{P}\{(u^1, u^2) \in S(m, m)\} \\ & \quad + (e^{-2\gamma T} + \delta + \delta^2) \mathbb{P}\{(u^1, u^2) \in R(k-2)\} \\ & \quad + (2\delta^2 + e^{-2\gamma T}) \mathbb{P}\{(u^1, u^2) \in S(k-2, k-2)\} \\ & \leq f_1 e^{-\gamma k T} + f_2 \delta^{k-1} + f_3 \delta^k \\ & \leq K \kappa^k, \end{aligned}$$

where  $f_i, i = 1, 2, 3$ , increases on  $k$  at most polynomially.

For  $S(k, k)$ , we have

$$\begin{aligned} & \mathbb{P}\{(u^1, u^2) \in S(k, k)\} \\ & \leq e^{-\gamma k T} \sum_{m=0}^{k-2} e^{\gamma m T} \mathbb{P}\{(u^1, u^2) \in S(m, k-1)\} \\ & \quad + \delta \mathbb{P}\{(u^1, u^2) \in S(k-1, k-1)\} + \mathbb{P}\{(u^1, u^2) \in R(k-1)\}. \end{aligned}$$

By the same analysis for  $R(k)$ , we also have

$$\mathbb{P}\{(u^1, u^2) \in S(k, k)\} \leq K \kappa^k.$$

This completes the proof.  $\square$

### 5. Ergodicity of the local system

In this section, we prove the existence and uniqueness of invariant measure, and then the ergodicity in the LSSH system.

Let  $\mathcal{P}(H)$  be the metric space of all probability measures on  $(H, \mathcal{B}(H))$  endowed with the metric  $\|\cdot\|_L^*$  defined by

$$\|\mu - \nu\|_L^* = \sup \left\{ \int_H f \, d(\mu - \nu) : \|f\|_L \leq 1 \right\}, \quad \mu, \nu \in \mathcal{P}(H),$$

where  $f$  is a measurable function on  $H$  and

$$\|f\|_L = \sup_x |f(x)| + \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} : x \neq y \in H \right\}.$$

It is well known that  $\|\cdot\|_L^*$  generates the weak topology and  $\mathcal{P}(H)$  is a complete space under  $\|\cdot\|_L^*$ . We define the semigroup  $S(t)$  on  $\mathcal{P}(H)$  generated by (6)–(8) as

$$S(t)\mu(\Gamma) = \int_H P(t, x, \Gamma)\mu(dx), \quad \Gamma \in \mathcal{B}(H).$$

A measure  $\mu$  on  $H$  is an invariant measure if

$$\mu(\Gamma) = S(t)\mu(\Gamma) = \int_H P(t, x, \Gamma)\mu(dx), \quad \Gamma \in \mathcal{B}(H).$$

We restrict  $S(t)$  on the set  $\mathcal{P}_2(H)$ , which is defined as

$$\mathcal{P}_2(H) = \left\{ \mu \in \mathcal{P}(H) : \int_H |z|^2 \mu(dz) < \infty \right\}.$$

Then we have the following proposition.

**Proposition 5.1.**  $S(t)$  maps  $\mathcal{P}_2(H)$  into  $\mathcal{P}_2(H)$ .

**Proof.** For any  $\mu \in \mathcal{P}_2(H)$  and  $t > 0$ , Lemma 3.2 yields

$$\begin{aligned} \int_H |y|^2 S(t)\mu(dy) &= \int_H \left( \int_H |y|^2 P(t, z, dy) \right) \mu(dz) \\ &= \int_H \mathbf{E}|u(t, z)|^2 \mu(dz) \\ &\leq e^{-\alpha t} \int_H |z|^2 \mu(dz) + R \\ &< \infty. \end{aligned}$$

This completes the proof.  $\square$

Now we can draw the following main results.

**Theorem 5.2.** *If (1) and (9) hold with  $N$  large enough, there is a positive constant  $\chi \in (0, 1)$  such that for any  $\mu_1, \mu_2 \in \mathcal{P}_2(H)$ ,*

$$\|S(t)\mu_1 - S(t)\mu_2\|_L^* \leq M(\mu_1, \mu_2)\chi^t, \quad t \geq 0.$$

Here  $M(\mu_1, \mu_2)$  only depends on  $\mu_1$  and  $\mu_2$ .

**Proof.** We follow the proof of [21]. Arbitrarily fixed  $t > 0$  and let  $k = k(t)$  be the smallest integer such that  $t \leq k(t)T$ , where  $T$  is the constant in Section 4, and let  $u_0^i, i = 1, 2$ , be

random variables in  $H$  with distribution  $\mu_i$ . Let  $u^1(t)$  and  $u^2(t)$  be the weak solutions of (6)–(8) on  $[0, kT]$ . Then we just need to show that

$$p(t) = \mathbb{P}\{|u^1(t) - u^2(t)| > C_1 e^{-\sigma_1 t}\} \leq C_2 e^{-\sigma_2 t}, \tag{19}$$

where  $C_i, i = 1, 2$ , are positive constants only depending on the initial functions and  $\sigma_i, i = 1, 2$ , are some positive constants.

We consider the coupling solution defined in Section 4. Define the following event:

$$G(k) = \left\{ (u_k^1, u_k^2) \in \bigcup_{m=0}^{[ck]} S(m, k) \right\},$$

where  $0 < c < 1$  is some constant. Clearly,

$$p(t) \leq \mathbb{P}(G(k)^c) + \mathbb{P}(G(k) \cap \{|u^1 - u^2| > C_1 e^{-\sigma_1 t}\}). \tag{20}$$

We claim that

$$\mathbb{P}(G(k)^c) \leq C_2 e^{-\sigma_2 t}, \tag{21}$$

$$\mathbb{P}(G(k) \cap \{|u^1 - u^2| > C_1 e^{-\sigma_1 t}\}) = 0. \tag{22}$$

First, we prove (24). Since all the orbits in  $S(m, k)$  come from  $R(m) \cup S(m, m)$ , we have

$$G(k)^c \subset \bigcup_{m=[ck]+1}^k S(m, k) \cup R(k) \subset \bigcup_{m=[ck]+1}^k R(m) \cup S(m, m).$$

Now by Proposition 4.5, we get

$$\begin{aligned} \mathbb{P}(G(k)^c) &\leq \sum_{m=[ck]+1}^k K \kappa^m \leq K \kappa^{ck} (k - [ck]) \\ &\leq \tilde{K} \tilde{\kappa}^k \end{aligned}$$

for some proper  $\tilde{\kappa} \in (0, 1)$  and  $\tilde{K}$ . Since  $k \geq \frac{t}{T}$ , we have

$$\mathbb{P}(G(k)^c) \leq C_2 e^{-\sigma_2 t}$$

where  $\sigma_2 = -T^{-1} \ln \tilde{\kappa}$  and  $C_2 = \tilde{K}$ .

Next we prove (25). It is enough to prove that if  $(u^1, u^2) \in G(k)$  and  $C_1$  is large enough

$$|u^1 - u^2| \leq C_1 e^{-\sigma t}. \tag{23}$$

Indeed, by the definition of  $G(k)$  for  $(u^1, u^2) \in G(k)$ , there is an integer  $l, 0 \leq l \leq [ck]$ , such that  $(u^1, u^2) \in S(l, k)$ . Therefore, relations (12)–(14) are satisfied. Then as Proposition 2.1, for  $u = u_1 - u_2$ , we have

$$|u(t)| = |\rho(t)| \leq 2de^{-\lambda(t-lT)}.$$

Notice that  $lT \leq ckT \leq c(t+T)$ , then  $t-lT \geq (1-c)t - cT$ . Hence,  $|u(t)| \leq 2de^{\lambda cT} e^{-\lambda(1-c)t}$ . Thus (25) holds with  $C_1 = 2de^{\lambda cT}$  and  $\sigma_1 = \lambda(1-c)$ . Finally, taking  $M(\mu_1, \mu_2) = C_2$  and  $\chi = e^{-\sigma_2}$ , we obtain the conclusion and the proof is completed.  $\square$

**Corollary 5.3.** *If enough modes are forced and condition (1) holds, then the LSSH system has a unique invariant measure  $\mu_0$ , such that for any  $u \in H$  and  $t > 0$ ,*

$$\|P(t, u, \cdot) - \mu_0\|_L^* \leq M(|u|^2)\chi^t,$$

where  $M(|u|^2)$  is a positive constant and  $\chi$  is as in Theorem 5.2.

### 6. Ergodicity of the nonlocal system

In this section, we turn to the following NLSSH system on the real line:

$$u_t - qu + (1 + \partial_{xx})^2 u + uG * u^2 = \tilde{F}(x, t), \tag{24}$$

$$u(0) = u_0, \tag{25}$$

$$u(t, x) = u(t, x + 2\pi), \tag{26}$$

where

$$G * u^2 = \int_0^{2\pi} G(x - \xi)u^2(\xi, t) d\xi.$$

We always assume that

$$\tilde{b}_i \neq 0, \quad 1 \leq i \leq \tilde{N} \leq N'', \tag{27}$$

$$\tilde{B}_0 = \sum_{i=1}^{N''} \tilde{b}_i^2 < \infty. \tag{28}$$

First, we consider the following case: for every  $x \in \mathbb{R}$

$$0 < b \leq G(x) \leq a, \tag{29}$$

where  $a$  and  $b$  are the positive constants (i.e.,  $G > 0$  is a positive kernel). Then we have the same energy estimation as LSSH system.

**Lemma 6.1.** *There is a positive constant  $\tilde{C}_1$ , such that for any  $r > 0$ ,*

$$\mathbb{P} \left\{ |u(t)|^2 + \int_0^t |u(s)|^2 ds \leq |u(0)|^2 + \tilde{C}_1 t + r, \text{ for } t \geq 0 \right\} \geq 1 - e^{-r}.$$

**Proof.** We only estimate the nonlinear term. In fact,

$$\langle uG * u^2, u \rangle = \langle G * u^2, u^2 \rangle \geq b|u|^4.$$

Set  $\tilde{C}_1 = \max_x \{2(q + \tilde{b}_{\max}^2 + \frac{1}{2})x^2 - bx^4 + \tilde{B}_0\}$ , where

$$\tilde{b}_{\max} = \max_i \tilde{b}_i. \tag{30}$$

By using the same analysis as in Lemma 3.1, we can finish the proof.  $\square$

Note that, for the nonlocal nonlinearity, only the part of results in Proposition 2.1 still holds. But this does not affect the coupling result. Taking the same decomposition of NLSSH system as one of LSSH system, we have

**Proposition 6.2.** *For any  $u_0^1, u_0^2 \in H$ , if  $l_{m,k}^1 = l_{m,k}^2$  and  $u^1, u^2$  satisfy (14), then there exists a constant  $\tilde{\lambda} > 0$  such that*

$$|u^1 - u^2| \leq e^{-\tilde{\lambda}(t-mT)} |u^1(mT) - u^2(mT)|, \quad t \in [mT, kT].$$

**Proof.** Since  $l_{m,k}^1 = l_{m,k}^2$ , we only consider the high mode. Let  $\rho = h^1 - h^2$ . Then,  $\rho$  satisfies

$$\dot{\rho} = A\rho + \varrho\rho - P_h(u^1 G * (u^1)^2 - u^2 g * (u^2)^2).$$

Taking the scalar product of the above equation with  $\rho$  in  $H_h$  and notice that

$$\begin{aligned} & \langle u^1 G * (u^1)^2 - u^2 G * (u^2)^2, \rho \rangle \\ &= \langle u^1 G * (u^1)^2 - u^1 G * (u^2)^2 + u^1 G * (u^2)^2 - u^2 G * (u^2)^2, \rho \rangle \\ &= \langle u^1 G * ((u^1)^2 - (u^2)^2), \rho \rangle + \langle (u^1 - u^2) G * (u^2)^2, \rho \rangle \\ &\geq \langle u^1 G * (u^1 - u^2)(u^1 + u^2), \rho \rangle, \end{aligned}$$

we have

$$\begin{aligned} \partial_t |\rho|^2 &\leq (\alpha_{\tilde{N}} + \varrho) |\rho|^2 + |\langle u^1 G * (u^1 - u^2)(u^1 + u^2), \rho \rangle|, \\ &\leq (\alpha_{\tilde{N}} + \varrho) |\rho|^2 + a \left( \frac{3}{2} |u^1|^2 + |u^2|^2 \right) |\rho|^2, \end{aligned}$$

where  $\alpha_{\tilde{N}}$  is the  $\tilde{N}$ th eigenvalue of  $A$ . Then

$$\begin{aligned} |\rho|^2 &\leq |\rho(mT)|^2 \exp \left\{ \alpha_{\tilde{N}} + \varrho + a \int_{mT}^t \left( \frac{3}{2} |u^1|^2 + |u^2|^2 \right) ds \right\} \\ &\leq |\rho(mT)|^2 \exp \left\{ \left( \alpha_{\tilde{N}} + \varrho + \frac{a}{t - mT} \int_{mT}^t \left( \frac{3}{2} |u^1|^2 + |u^2|^2 \right) ds \right) (t - mT) \right\} \\ &\leq |\rho(mT)|^2 \exp \left\{ \left( \alpha_{\tilde{N}} + \varrho + \frac{5}{2} a \tilde{C}_1 + a \right) (t - mT) \right\}, \end{aligned}$$

since we can always take  $r < T$ . If we choose  $\tilde{N}$  so large that  $\alpha_{\tilde{N}} + \varrho + \frac{5}{2} a \tilde{C}_1 + a < 0$ , then

$$|\rho|^2 \leq e^{-\tilde{\lambda}(t-mT)} |\rho(mT)|, \quad t \in [mT, kT],$$

for some constant  $\tilde{\lambda}$ . This completes the proof.  $\square$

**Remark 6.3.** The above proposition is weaker than Proposition 2.1, but it is enough for the coupling result in Lemma 4.1.

We have to give more estimation for the coupling result. For Lemma 4.1, we have to estimate  $|\tilde{B}(l, h_0^1, h_0^2)| = |P_l u^1 G * (u^1)^2 - P_l u^2 G * (u^2)^2|$ . In fact,

$$\begin{aligned} &|u^1 G * (u^1)^2 - u^2 G * (u^2)^2| \\ &= |(l + h^1)G * (l + h^1)^2 - (l + h^2)G * (l + h^2)^2| \\ &\leq |lG * [(l + h^1)^2 - (l + h^2)^2]| \\ &\quad + |h^1 G * (l^2 + (h^1)^2 + 2lh^1) - h^2 G * (l^2 + (h^2)^2 + 2lh^2)| \\ &\leq |2lG * [l(h^1 - h^2)]| + |lG * [(h^1 - h^2)(h^1 + h^2)]| \\ &\quad + |(h^1 - h^2)G * l^2| + |h^1 G * (h^1)^2 - h^2 G * (h^2)^2| \\ &\quad + 2|h^1 G * (lh^1) - h^2 G * (lh^2)|. \end{aligned}$$

From (29) we have

$$|2lG * [l(h^1 - h^2)]| \leq 2a|h^1 - h^2||l|^2, \tag{31}$$

$$|lG * [(h^1 - h^2)(h^1 + h^2)]| \leq a|h^1 - h^2||h^1 + h^2||l|, \tag{32}$$

$$|(h^1 - h^2)G * l^2| \leq a|h^1 - h^2||l|^2, \tag{33}$$

$$|h^1 G * (h^1)^2 - h^2 G * (h^2)^2| \tag{34}$$

$$\leq |h^1 G * (h^1)^2 - h^2 G * (h^1)^2| + |h^2 G * (h^1)^2 - h^2 G * (h^2)^2| \tag{35}$$

$$\leq a|h^1 - h^2||h^1|^2 + a|h^1 - h^2||h^1 + h^2||h^2|. \tag{36}$$

Similarly, we have

$$2|h^1 G * (lh^1) - h^2 G * (lh^2)| \leq 2a|h^1 - h^2||h^1 + h^2||l|. \tag{37}$$

Then, from (33)–(39) and  $|u|^2$  increases polynomially, we can derive the coupling result in Lemma 4.1 by Proposition 6.2.

For Lemma 4.3, we have to estimate  $|\tilde{B}| = |v\tilde{l} + P_l u G * (l + h)^2|$ . Since

$$|uG * u^2| \leq a|u|^3, \tag{38}$$

$|\tilde{B}|$  is bounded.

With all the above analysis, we draw the following conclusion.

**Theorem 6.4.** *If (27) and (28) hold with  $\tilde{N}$  large enough, the NLSSH system with the positive kernel (29) has a unique invariant measure, and it is exponentially attracted in  $\mathcal{P}_2(H)$ .*

The above analysis for the positive kernel (29) does not work for the nonnegative kernels. But we can still work for a special nonnegative kernel. See [23].

Define

$$J(x) = \begin{cases} c \exp\left(-\frac{1}{1-x^2}\right) & \text{if } x < 1, \\ 0 & \text{if } x \geq 1. \end{cases} \tag{39}$$

where

$$c = \left( \int_0^1 \exp\left(-\frac{1}{1-x^2}\right) dx \right)^{-1}.$$

We further define that for  $\delta > 0$ ,

$$J_\delta(x) = \delta^{-2} J\left(\frac{x}{\delta}\right).$$

Let  $C_0(\bar{I})$  be the space of continuous functions with compact support in  $I$ . It is known that [12]

$$\|J_\delta * f - f\|_{C_0(\bar{I})} \rightarrow 0 \quad \text{as } \delta \rightarrow 0 \tag{40}$$

for any  $f \in C_0(\bar{I})$ . Thus for any  $\varepsilon > 0$ , there is a  $\delta_0 = \delta_0(\varepsilon) > 0$ , such that

$$J_{\delta_0} * f \geq f - \varepsilon. \tag{41}$$

We consider a special kernel  $G(x) = J_{\delta_0}(x)$ , with  $\delta_0 = \delta_0(\varepsilon)$  as in (41), for the NLSSH (3). Then,  $G$  satisfies

$$0 \leq G \leq \frac{c}{\delta_0^2}. \tag{42}$$

Note that the lower bound of  $G$  is  $b = 0$ , so some estimation above does not apply. But due to (41) we can still get the similar energy estimates. And notice that we only use the lower bound in the energy estimates, so we can have the same ergodic result.

From [18,31] or [23], we know that problem (24)–(26) has a unique solution  $u(t, x, u_0) \in C([0, T]; H) \cap L^\infty((0, T), H_0^2(I))$  and

$$u(t, x, u_0) \in C_0(\bar{I}), \forall t \geq t_0 > 0.$$

Thus (41) implies that

$$G(x) * u^2 \geq u^2 - \varepsilon. \tag{43}$$

Then the estimates of Lemma 6.1 still holds with

$$\tilde{C}_1 = \max_x \left\{ 2 \left( \mu + \tilde{b}_{\max}^2 + \frac{1}{2} + \varepsilon \right) x^2 - \frac{1}{2\pi} x^4 + \tilde{B}_0 \right\}. \tag{44}$$

The other estimates does not depends on the lower bound of  $G$ . Then we can derive the following result.

**Theorem 6.5.** *If (27) and (28) hold with  $\tilde{N}$  large enough, the NLSSH system with the nonnegative kernel  $J_{\delta_0}(x)$  has a unique invariant measure, and it is exponentially attracted in  $\mathcal{P}_2(H)$ .*

### 7. Dynamical difference between local and nonlocal systems

With the results on ergodicity of the local and nonlocal stochastic Swift–Hohenberg equations in previous sections, we can now compare some dynamical behavior.

From Proposition 2.1, we see that the local stochastic Swift–Hohenberg system is ergodic provided  $\alpha_N + \varrho < 0$ . Note that  $\alpha_N = -(N^2 - 1)^2$ . So we only need the number  $N$  of randomly forced modes to satisfy

$$N > \sqrt{\sqrt{\varrho} + 1}.$$

Note that the number  $N$  does not depend on the random forcing term, i.e., it does not depend on the coefficients  $b'_i s$  in the random forcing in (2).

However, from Proposition 6.2 the nonlocal stochastic Swift–Hohenberg system with positive kernel ( $0 < b \leq G(x) \leq a$ ) is ergodic provided that  $\tilde{N}$  satisfies  $\alpha_{\tilde{N}} + \varrho + \frac{5}{2}a\tilde{C}_1 + a < 0$ , where  $\tilde{C}_1 = \tilde{B}_0 + \frac{(\varrho + \tilde{b}_{\max}^2 + \frac{1}{2})^2}{b}$ , with  $\tilde{B}_0$  defined in (28) and  $\tilde{b}_{\max}$  defined in (30). Thus we only need the number of randomly forced modes to satisfy

$$\tilde{N} > \sqrt{\sqrt{\varrho + a + \frac{5}{2}a(\tilde{B}_0 + \frac{(\varrho + \tilde{b}_{\max}^2 + \frac{1}{2})^2}{b})} + 1}.$$

For the nonnegative kernel  $G(x)$  with upper bound  $a$ , we only need

$$\tilde{N} > \sqrt{\sqrt{\varrho + a + \frac{5}{2}a(\tilde{B}_0 + 2\pi(\varrho + \tilde{b}_{\max}^2 + \frac{1}{2} + \varepsilon)^2)} + 1}.$$

Clearly, we have the following comparison between the local and nonlocal stochastic Swift–Hohenberg models: (i) The number of Fourier modes to be randomly excited for ensuring ergodicity of the *local* stochastic Swift–Hohenberg system depends only on the Rayleigh number through  $\varrho > 0$ , which measures the difference of the Rayleigh number from its critical convection onset value; (ii) The number of Fourier modes to be randomly excited for ensuring ergodicity of the *nonlocal* stochastic Swift–Hohenberg system depends on the bound of the kernel  $G$  in the nonlocal term, and the random term itself, as well as on the Rayleigh number.

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